RG Methods in Statistical Field Theory: Quiz 8 Solution

Consider a quantum system of two sites, with one fermionic state at each site. The creation/destruction operators for this state are $c_1^\dagger$, c_1, and $c_2^\dagger$, c_2 for sites 1 and 2 respectively. We have a Hamiltonian:

$$\mathcal{H} = -t(c_1^\dagger c_2 + c_2^\dagger c_1) - \mu(c_1^\dagger c_1 + c_2^\dagger c_2)$$

The first term describes hopping between the sites, while the second term is the chemical potential. In the Fock space there are four basis states, which we label $|n_1, n_2\rangle$, where n_i is the occupation number of site i:

$$|0,0\rangle, \quad |1,0\rangle = c_1^\dagger|0,0\rangle, \quad |0,1\rangle = c_2^\dagger|0,0\rangle, \quad |1,1\rangle = c_1^\dagger c_2^\dagger|0,0\rangle,$$

Find the partition function Z by any method you choose. *Hint:* The easiest way is to construct the Hamiltonian matrix in the Fock space basis, and then find its eigenvalues.

<u>Answer:</u> The operator $\mathcal{H}$ acting on the basis states gives the following results:

$$\mathcal{H}|0,0\rangle = 0, \quad \mathcal{H}|1,0\rangle = -t|0,1\rangle - \mu|1,0\rangle, \quad \mathcal{H}|0,1\rangle = -t|1,0\rangle - \mu|0,1\rangle, \quad \mathcal{H}|1,1\rangle = -2\mu|1,1\rangle$$

Thus $\mathcal{H}$ corresponds to the following matrix (with the rows and columns in the same order as the basis states are listed above):

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mu & -t & 0 \\ 0 & -t & -\mu & 0 \\ 0 & 0 & 0 & -2\mu \end{pmatrix}$$

The eigenvalues of this matrix (the eigenenergies E_n of the system) are: 0, $-\mu - t$, $-\mu + t$, -2μ. Thus the partition function is given by:

$$Z = \sum_n e^{-\beta E_n} = 1 + e^{\beta(\mu+t)} + e^{\beta(\mu-t)} + e^{2\beta\mu}$$