RG Methods in Statistical Field Theory: Quiz 6 Solution

Friday, November 10, 2006

Consider the d-dimensional, $n = 1$ Gaussian model, described by the Hamiltonian:

$$\mathcal{H} = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{2}(r + cq^2 + Lq^4 + \cdots)m(\mathbf{q})m(-\mathbf{q})$$

(a) Write down the renormalization group equations for r', c', and L', choosing the constant ζ such that $c' = c$. Here ζ is the constant of proportionality in the definition: $m'(\mathbf{q}') \equiv \zeta^{-1}m_<(\mathbf{q})$, where $\mathbf{q}' = b\mathbf{q}$.

__**Answer:**__ Using the same argument as in lecture, we can show that $Z = C_0 \int \mathcal{D}m_< \exp(-\beta\mathcal{H}_<)$, where C_0 is a constant that comes from integrating out the fast modes. The slow-mode part of the Hamiltonian is given by:

$$\begin{aligned}\mathcal{H}_< &= \int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{2}(r + cq^2 + Lq^4 + \cdots)m_<(\mathbf{q})m_<(-\mathbf{q}) \\ &= \int_0^\Lambda b^{-d}\frac{d^d\mathbf{q}'}{(2\pi)^d}\frac{1}{2}(r + cb^{-2}q'^2 + Lb^{-4}q'^4 + \cdots)\zeta^2 m'(\mathbf{q}')m'(-\mathbf{q}')\end{aligned}$$

Thus $r' = b^{-d}\zeta^2 r$, $c' = b^{-d-2}\zeta^2 c$, and $L' = b^{-d-4}\zeta^2 L$. To make $c' = c$ we choose $\zeta = b^{(d+2)/2}$.

(b) The renormalization group equations of part (a) have a fixed point at $r^* = L^* = \cdots = 0$. Let us see how various terms added to the Hamiltonian behave near this fixed point. Consider a general term of the following type:

$$u\int d^d\mathbf{x}\, m^p(\mathbf{x}) = u\int_0^\Lambda d^d\mathbf{q}_1 \cdots \int_0^\Lambda d^d\mathbf{q}_{p-1}\, m(\mathbf{q}_1)m(\mathbf{q}_2)\cdots m(\mathbf{q}_{p-1})m(-\mathbf{q}_1 - \mathbf{q}_2 - \cdots - \mathbf{q}_{p-1})$$

where $p \geq 4$ is an even integer, and u is a small constant. Since the $\mathbf{q}$ integrals go from 0 to Λ, this term mixes fast and slow modes, making it difficult to calculate the RG transformation (we will see how this is done in today's lecture). To get some idea of how this term behaves under RG, we can make an easy, crude approximation: make the upper limit of each integral Λ/b. (Essentially we are ignoring any contribution that involves fast modes.) Now use the relations $m'(\mathbf{q}') = \zeta^{-1}m_<(\mathbf{q})$ and $\mathbf{q}' = b\mathbf{q}$ to rewrite the term and find the RG equation for u'. Keep the constant ζ the same as in part (a).

__**Answer:**__ We use the same method as in part (a) to rewrite the u term. Here there are $p - 1$ factors of the form $d^d\mathbf{q}_i$, and p Fourier-transformed variables $m(\mathbf{q}_i)$. Thus we find:

$$u' = b^{-d(p-1)}\zeta^p u = b^{-d(p-1)}b^{p(d+2)/2}u = b^{(-dp+2d+2p)/2}u$$

(c) Write the RG equation of part (b) in the form: $u' = b^{\lambda_u}u$ for some exponent λ_u. Check that for $d < d_c$ the u direction is relevant, and for $d > d_c$ the u direction is irrelevant. Find d_c.

__**Answer:**__ From part (b) we see that $\lambda_u = (-dp + 2d + 2p)/2$. For $d > 2p/(p-2)$ the exponent $\lambda_u < 0$, and for $d < 2p/(p-2)$ we have $\lambda_u > 0$. Thus:

$$d_c = 2p/(p-2)$$