RG Methods in Statistical Field Theory: Quiz 3 Solution

Friday, October 13, 2006

A superfluid system can be described using an order parameter which is a complex quantum wavefunction $\psi(\mathbf{x})$. The Hamiltonian functional for a superfluid (in $d = 3$ dimensions) is given by:

$$\mathcal{H} = \int d^3\mathbf{x} \left[\frac{r}{2} |\psi(\mathbf{x})|^2 + u|\psi(\mathbf{x})|^4 + \frac{c}{2} (\nabla\psi) \cdot (\nabla\psi^*) \right] ,$$

where r, u, and c are real constants. The mean-field solution which minimizes this Hamiltonian can be written as $\psi(\mathbf{x}) = \psi_0$, where ψ_0 is a *real* number independent of position. Let us calculate the energy of phase fluctuations around this mean-field solution. Plug the following form of $\psi(\mathbf{x})$ into the Hamiltonian:

$$\psi(\mathbf{x}) = \psi_0 e^{i\theta(\mathbf{x})} ,$$

where $\theta(\mathbf{x})$ is some small phase that varies over space. Show that the Hamiltonian can be written as:

$$\mathcal{H} = \mathcal{H}_0 + \int d^3\mathbf{x} \frac{\rho_s}{2} (\nabla\theta)^2 .$$

Find $\mathcal{H}_0$ and ρ_s in terms of r, u, c, ψ_0, and V, the volume of the system.

Answer: Let us plug $\psi(\mathbf{x}) = \psi_0 e^{i\theta(\mathbf{x})}$ into each term in the Hamiltonian:

$$\begin{aligned}
\frac{r}{2} |\psi(\mathbf{x})|^2 &= \frac{r}{2} \psi_0^2 \\
u|\psi(\mathbf{x})|^4 &= u\psi_0^4 \\
\frac{c}{2} (\nabla\psi) \cdot (\nabla\psi^*) &= \frac{c}{2} (i\psi_0 e^{i\theta(\mathbf{x})} \nabla\theta(\mathbf{x})) \cdot (-i\psi_0 e^{-i\theta(\mathbf{x})} \nabla\theta(\mathbf{x})) \\
&= \frac{c\psi_0^2}{2} (\nabla\theta(\mathbf{x}))^2
\end{aligned}$$

Putting this all together we get:

$$\begin{aligned}
\mathcal{H} &= \int d^3\mathbf{x} \left[\frac{r}{2} \psi_0^2 + u\psi_0^4 + \frac{c\psi_0^2}{2} (\nabla\theta)^2 \right] \\
&= V \left(\frac{r}{2} \psi_0^2 + u\psi_0^4 \right) + \int d^3\mathbf{x} \frac{c\psi_0^2}{2} (\nabla\theta)^2
\end{aligned}$$

Thus:

$$\begin{aligned}
\mathcal{H}_0 &= V \left(\frac{r}{2} \psi_0^2 + u\psi_0^4 \right) \\
\rho_s &= c\psi_0^2
\end{aligned}$$