RG Methods in Statistical Field Theory: Quiz 2 Solution

Friday, October 6, 2006

Consider a d-dimensional system with an n-component local magnetization $\langle \mathbf{m}(\mathbf{x}) \rangle$. The magnetization at position $\mathbf{x}$ in principle depends on the magnetic field applied at all positions $\mathbf{x}'$ in the system. In other words, we can consider $\langle \mathbf{m}(\mathbf{x}) \rangle$ to be a functional of $\mathbf{h}(\mathbf{x}')$.

(a) Assume that $\langle \mathbf{m}(\mathbf{x}) \rangle$ equals some function $\mathbf{m}^0(\mathbf{x})$ when $\mathbf{h}(\mathbf{x}') = 0$ for all $\mathbf{x}'$. Now add a small, spatially-varying magnetic field $\delta \mathbf{h}(\mathbf{x}')$ throughout the system. Write a functional Taylor series expansion of $\langle m_i(\mathbf{x}) \rangle$ around $m_i^0(\mathbf{x})$, going up to first order in $\delta \mathbf{h}(\mathbf{x}')$. [Here $\langle m_i(\mathbf{x}) \rangle$ is the ith component of $\langle \mathbf{m}(\mathbf{x}) \rangle$.]

<u>**Answer:**</u>

$$\langle m_i(\mathbf{x}) \rangle = m_i^0(\mathbf{x}) + \int d^d\mathbf{x}' \, \frac{\delta \langle m_i(\mathbf{x}) \rangle}{\delta h_j(\mathbf{x}')} \delta h_j(\mathbf{x}') + \cdots$$

Here the Einstein summation convention is used, so that the repeated index j is summed over.

(b) Starting with the result from (a), plug in the definition of the nonlocal susceptibility $\chi_{ij}(\mathbf{x}, \mathbf{x}')$. For a small magnetic field along the jth direction of the form

$$\delta \mathbf{h}(\mathbf{x}') = H e^{-i\mathbf{q}\cdot\mathbf{x}'} \hat{\mathbf{e}}_j \, ,$$

show that:

$$\langle m_i(\mathbf{x}) \rangle = m_i^0(\mathbf{x}) + H e^{-i\mathbf{q}\cdot\mathbf{x}} \chi_{ij}(\mathbf{q}) + \cdots \, ,$$

where $\chi_{ij}(\mathbf{q})$ is the Fourier transform of $\chi_{ij}(\mathbf{x}, \mathbf{x}')$. [By translational invariance, you can assume that $\chi_{ij}(\mathbf{x}, \mathbf{x}') = \chi_{ij}(\mathbf{x}' - \mathbf{x})$.]

This result shows that $\chi_{ij}(\mathbf{q})$ measures the change in the ith component of the magnetization when we apply a small periodic magnetic field of wavevector $\mathbf{q}$ along the jth direction.

<u>**Answer:**</u> The definition of the nonlocal susceptibility is:

$$\chi_{ij}(\mathbf{x}, \mathbf{x}') = \frac{\delta \langle m_i(\mathbf{x}) \rangle}{\delta h_j(\mathbf{x}')} \, .$$

Plugging this into the result from (a), together with $\delta \mathbf{h}(\mathbf{x}') = H e^{-i\mathbf{q}\cdot\mathbf{x}'} \hat{\mathbf{e}}_j$, we find:

$$\begin{aligned}
\langle m_i(\mathbf{x}) \rangle &= m_i^0(\mathbf{x}) + H \int d^d\mathbf{x}' \, \chi_{ij}(\mathbf{x}, \mathbf{x}') e^{-i\mathbf{q}\cdot\mathbf{x}'} \\
&= m_i^0(\mathbf{x}) + H e^{-i\mathbf{q}\cdot\mathbf{x}} \int d^d\mathbf{x}' \, \chi_{ij}(\mathbf{x}' - \mathbf{x}) e^{-i\mathbf{q}\cdot(\mathbf{x}' - \mathbf{x})} \\
&= m_i^0(\mathbf{x}) + H e^{-i\mathbf{q}\cdot\mathbf{x}} \chi_{ij}(\mathbf{q}) \, .
\end{aligned}$$