RG Methods in Statistical Field Theory: Quiz 1 Solution

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Consider a one-dimensional lattice of sites x_α, where the lattice spacing $x_{\alpha+1} - x_\alpha = \ell$. At each site x_α we have a quantity ϕ_α that is a continuous variable ranging between $-\infty$ and ∞. The Hamiltonian for this system is:

$$\mathcal{H} = \sum_\alpha \left[A(\cosh(\phi_\alpha) - 1) + \frac{K}{2}(\phi_{\alpha+1} - \phi_\alpha)^2 \right] ,$$

where the constants A, $K > 0$.

The partition function for this model is given by:

$$Z = \int_{-\infty}^{\infty} \prod_\alpha d\phi_\alpha \exp\left(-\beta\mathcal{H}\right) .$$

In the continuum limit we can write each ϕ_α as $\phi_\alpha = \phi(x_\alpha)$, where $\phi(x)$ is a continuous function of x. In this limit the partition function becomes (to lowest order) the functional integral:

$$Z = \int \mathcal{D}\phi(x)\ \exp\left(-\beta \int dx\ \left[\frac{r}{2}\phi^2(x) + u\phi^4(x) + \frac{c}{2}\left(\frac{\partial}{\partial x}\phi(x)\right)^2 + \cdots \right] \right) .$$

Find the coupling constants r, u, and c in terms of A, K and ℓ. *Hint:* Write out $\phi(x_{\alpha+1})$ as a Taylor series around $\phi(x_\alpha)$. Also, use the fact that $\cosh(x) = 1 + x^2/2 + x^4/24 + \cdots$ for small x.

Answer: We expand $\phi(x_{\alpha+1}) = \phi(x_\alpha + \ell)$ as a Taylor series around $\phi(x_\alpha)$:

$$\phi(x_\alpha + \ell) = \phi(x_\alpha) + \ell\frac{\partial}{\partial x}\phi(x_\alpha) + \cdots .$$

Plugging this into the Hamiltonian, and using the Taylor expansion for $\cosh(x)$, we find:

$$\mathcal{H} = \sum_\alpha \left[\frac{A}{2}\phi^2(x_\alpha) + \frac{A}{24}\phi^4(x_\alpha) + \frac{K\ell^2}{2}\left(\frac{\partial}{\partial x}\phi(x_\alpha)\right)^2 + \cdots \right] .$$

We pull a factor of ℓ outside the brackets:

$$\mathcal{H} = \sum_\alpha \ell \left[\frac{A}{2\ell}\phi^2(x_\alpha) + \frac{A}{24\ell}\phi^4(x_\alpha) + \frac{K\ell}{2}\left(\frac{\partial}{\partial x}\phi(x_\alpha)\right)^2 + \cdots \right] .$$

In the continuum limit $\sum_\alpha \ell \to \int dx$ and $\phi(x_\alpha) \to \phi(x)$, so we have:

$$\mathcal{H} = \int dx\ \left[\frac{r}{2}\phi^2(x) + u\phi^4(x) + \frac{c}{2}\left(\frac{\partial}{\partial x}\phi(x)\right)^2 + \cdots \right] ,$$

where:

$$r = \frac{A}{\ell}\,, \quad u = \frac{A}{24\ell}\,, \quad c = K\ell\,.$$