RG Methods in Statistical Field Theory: Problem Set 7

due: Friday, November 17, 2006

In lecture we have looked at models where the order parameter $\mathbf{m}(\mathbf{x})$ is an n-component vector of arbitrary length. What happens when we restrict the order parameter to be a vector of fixed length? As we will see below, this leads to a field theory involving complicated interactions between the Goldstone modes, known as the *nonlinear σ-model*. Expanding the Hamiltonian through a perturbation series, we will be able to analyze this model through renormalization group techniques.

Let us begin with a magnetic system on a d-dimensional hypercubic lattice, where at each point $\mathbf{x}_\alpha$ there is an n-component spin $\mathbf{s}(\mathbf{x}_\alpha)$. The lattice spacing is ℓ, and the spins all have unit length, so that $|\mathbf{s}(\mathbf{x}_\alpha)| = 1$. The Hamiltonian $-\beta\mathcal{H}$ is given by:

$$-\beta\mathcal{H} = K \sum_{\langle\alpha\gamma\rangle} \mathbf{s}(\mathbf{x}_\alpha) \cdot \mathbf{s}(\mathbf{x}_\gamma)$$

where $K > 0$ and the sum $\langle\alpha\gamma\rangle$ is over nearest-neighbor sites in the lattice. Note that K is a dimensionless parameter: $K \equiv K_0/k_B T$ where the constant K_0 has units of energy. If we measure temperature T in units of K_0/k_B, then $T = K^{-1}$. The partition function is:

$$Z = \int_{-\infty}^{\infty} \prod_\alpha \left[d^n\mathbf{s}(\mathbf{x}_\alpha)\, \delta(\mathbf{s}^2(\mathbf{x}_\alpha) - 1) \right] e^{-\beta\mathcal{H}}$$

Here $\int_{-\infty}^{\infty} d^n\mathbf{s}(\mathbf{x}_\alpha) \equiv \int_{-\infty}^{\infty} ds_1(\mathbf{x}_\alpha)\, ds_2(\mathbf{x}_\alpha) \ldots ds_n(\mathbf{x}_\alpha)$ integrates each component of $\mathbf{s}(\mathbf{x}_\alpha)$ from $-\infty$ to ∞. To keep the length of $\mathbf{s}(\mathbf{x}_\alpha)$ fixed, we include the delta function $\delta(\mathbf{s}^2(\mathbf{x}_\alpha) - 1)$, where $\mathbf{s}^2(\mathbf{x}_\alpha) = \mathbf{s}(\mathbf{x}_\alpha) \cdot \mathbf{s}(\mathbf{x}_\alpha)$.

(a) Show that the Hamiltonian can be rewritten as:

$$-\beta\mathcal{H} = -\frac{K}{2} \sum_{\langle\alpha\gamma\rangle} (\mathbf{s}(\mathbf{x}_\alpha) - \mathbf{s}(\mathbf{x}_\gamma))^2 + C$$

where C is a constant. (We will ignore this constant term from now on.)

(b) At $T = 0$ the system is in a ground state where all the spins are aligned. Let us choose one particular ground state configuration, where all the spins point along the nth direction: $\mathbf{s}(\mathbf{x}_\alpha) = \{0, \ldots, 0, 1\}$ for all positions α. We are interested in constructing a theory for low temperatures, describing fluctuations around this ground state. Let us write the vector $\mathbf{s}(\mathbf{x}_\alpha)$ at low T as follows:

$$\mathbf{s}(\mathbf{x}_\alpha) = \{\pi_1(\mathbf{x}_\alpha), \pi_2(\mathbf{x}_\alpha), \ldots, \pi_{n-1}(\mathbf{x}_\alpha), \sigma(\mathbf{x}_\alpha)\}$$

Here $\pi_i(\mathbf{x}_\alpha)$ is a small transverse fluctuation along the ith direction, $i = 1, \ldots, n-1$. (These are the $n-1$ Goldstone modes that occur for a system with broken continuous symmetry.)

The variable $\sigma(\mathbf{x}_\alpha)$, describing the nth component of $\mathbf{s}(\mathbf{x}_\alpha)$, is not independent: the constraint that $\mathbf{s}^2(\mathbf{x}_\alpha) = 1$ means that $\sigma^2(\mathbf{x}_\alpha) = 1 - \boldsymbol{\pi}^2(\mathbf{x}_\alpha)$, where $\boldsymbol{\pi}(\mathbf{x}_\alpha)$ is the $n-1$ component vector whose ith component is $\pi_i(\mathbf{x}_\alpha)$. We can use this constraint (expressed through the delta functions) to simplify the partition function. Do the integral over $\sigma(\mathbf{x}_\alpha)$ at each α, and show that Z becomes:

$$Z = \int \prod_\alpha d^{n-1}\boldsymbol{\pi}(\mathbf{x}_\alpha) \exp\left(-\frac{K}{2} \sum_{\langle\alpha\gamma\rangle} (\boldsymbol{\pi}(\mathbf{x}_\alpha) - \boldsymbol{\pi}(\mathbf{x}_\gamma))^2 \right.$$
$$\left. -\frac{K}{2} \sum_{\langle\alpha\gamma\rangle} \left(\sqrt{1 - \boldsymbol{\pi}^2(\mathbf{x}_\alpha)} - \sqrt{1 - \boldsymbol{\pi}^2(\mathbf{x}_\gamma)} \right)^2 - \frac{1}{2} \sum_\alpha \ln(1 - \boldsymbol{\pi}^2(\mathbf{x}_\alpha)) \right)$$

where we have left out any constant terms. *Hint:* To deal with the delta functions $\delta(\mathbf{s}^2(\mathbf{x}_\alpha) - 1) = \delta(\sigma^2(\mathbf{x}_\alpha) + \boldsymbol{\pi}^2(\mathbf{x}_\alpha) - 1)$ we use the following identity: if $g(x)$ is a function of x with roots x_i where $g(x_i) = 0$, then

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{g'(x_i)}$$

Here $g'(x_i)$ is the derivative of $g(x)$ evaluated at x_i. The function $\sigma^2(\mathbf{x}_\alpha) + \boldsymbol{\pi}^2(\mathbf{x}_\alpha) - 1$ has roots at $\sigma(\mathbf{x}_\alpha) = \pm\sqrt{1 - \boldsymbol{\pi}^2(\mathbf{x}_\alpha)}$. Since we are assuming small transverse fluctuations $\boldsymbol{\pi}(\mathbf{x}_\alpha)$, a negative value for $\sigma(\mathbf{x}_\alpha)$ is unphysical, so we can ignore the negative root.

(c) Going to the continuum limit, we replace $\boldsymbol{\pi}(\mathbf{x}_\alpha)$ by a continuous function $\boldsymbol{\pi}(\mathbf{x})$ of position $\mathbf{x}$. Show that our partition function becomes the functional integral:

$$Z = \int \mathcal{D}\boldsymbol{\pi} e^{-\beta\mathcal{H}}$$

where

$$-\beta\mathcal{H} = -\frac{K}{2} \int d^d\mathbf{x} \left[(\nabla\boldsymbol{\pi}(\mathbf{x}))^2 + (\nabla\sqrt{1 - \boldsymbol{\pi}^2(\mathbf{x})})^2 \right] - \frac{\rho}{2} \int d^d\mathbf{x} \ln(1 - \boldsymbol{\pi}^2(\mathbf{x}))$$

Here $(\nabla\boldsymbol{\pi}(\mathbf{x}))^2 = \sum_{i=1}^d \sum_{j=1}^{n-1} \partial_i \pi_j(\mathbf{x}) \, \partial_i \pi_j(\mathbf{x})$, and the constant $\rho \equiv N/V = \ell^{-d}$ is the number of spins per unit volume. Note that the constant K here is different than that of parts (a) and (b): it has absorbed a factor of ℓ^{2-d}, so that now K has dimensions of $[\text{length}]^{2-d}$. *Hint:* Remember that in the continuum limit $\sum_\alpha = \ell^{-d} \sum_\alpha \ell^d = \ell^{-d} \int d^d\mathbf{x}$.

(d) The Hamiltonian found in part (c) describes complicated interactions between the Goldstone modes through its nonlinear terms. Before we tackle the full Hamiltonian, let us focus on the simple, Gaussian part:

$$-\beta\mathcal{H}_0 = -\frac{K}{2} \int d^d\mathbf{x} (\nabla\boldsymbol{\pi}(\mathbf{x}))^2 = -\frac{K}{2} \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \, q^2 \pi_i(\mathbf{q}) \pi_i(-\mathbf{q})$$

where we have also written it in terms of Fourier modes $\boldsymbol{\pi}(\mathbf{q})$, assuming a spherical Brillouin zone of radius Λ. Note that from this point on we will use the Einstein summation convention.

What is the average $\langle \pi_i(\mathbf{q}_1)\pi_j(\mathbf{q}_2)\rangle_0$ evaluated with respect to $\mathcal{H}_0$? (No calculations are necessary; don't forget the delta function!) Now calculate the single-site correlation function (i.e. the squared amplitude of the $\boldsymbol{\pi}$ fluctuations at $\mathbf{x}$):

$$\langle \boldsymbol{\pi}^2(\mathbf{x})\rangle_0 = \langle \pi_i(\mathbf{x})\pi_i(\mathbf{x})\rangle_0 = \int_0^\Lambda \frac{d^d\mathbf{q}_1}{(2\pi)^d}\int_0^\Lambda \frac{d^d\mathbf{q}_2}{(2\pi)^d}\langle \pi_i(\mathbf{q}_1)\pi_i(\mathbf{q}_2)\rangle_0 e^{i(\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{x}}$$

Show that the answer is:

$$\langle \boldsymbol{\pi}^2(\mathbf{x})\rangle_0 = \frac{(n-1)S_d}{(2\pi)^d K}\int_0^\Lambda dq\, q^{d-3}$$

Note that $\langle \boldsymbol{\pi}^2(\mathbf{x})\rangle_0$ is proportional to $K^{-1} = T$, and that for $d > 2$, the integral is convergent and equals $\Lambda^{d-2}/(d-2)$. But for $d \leq 2$, the integral diverges at the $q = 0$ limit. This is exactly the result you found in Problem Set 3: the long-wavelength Goldstone mode fluctuations destroy order for $T \neq 0$ in dimensions $d \leq 2$, so that order is only possible at $T = 0$. On the other hand, for $d > 2$, we have $\langle \boldsymbol{\pi}^2(\mathbf{x})\rangle_0 \propto T$. Thus for small T the fluctuations are small, and we expect that there is a nonzero transition temperature to an ordered phase. To check whether this expectation is true, let us now turn to the full Hamiltonian $-\beta\mathcal{H}$ and apply RG techniques.

(e) Since we are interested in low T behavior, it makes sense to make an expansion of $-\beta\mathcal{H}$ in terms of powers of T. How do we decide the order of T associated with a certain term? From part (d) above, we can assume that the magnitude of the fluctuations $\boldsymbol{\pi} \sim \mathcal{O}(T^{1/2})$. The parameter $K \sim \mathcal{O}(T^{-1})$. Thus the Gaussian part of the Hamiltonian:

$$-\beta\mathcal{H}_0 = -\frac{K}{2}\int d^d\mathbf{x}(\nabla\boldsymbol{\pi}(\mathbf{x}))^2 \sim \mathcal{O}(T^0)$$

Now expand the full Hamiltonian $-\beta\mathcal{H}$ in a Taylor series with respect to $\boldsymbol{\pi}(\mathbf{x})$, and keep terms up to $\mathcal{O}(T^1)$. Show that this gives:

$$\begin{aligned}-\beta\mathcal{H} &= -\beta\mathcal{H}_0 - \int d^d\mathbf{x}\left[\frac{K}{2}\left(\boldsymbol{\pi}(\mathbf{x})\cdot\nabla\boldsymbol{\pi}(\mathbf{x})\right)^2 - \frac{\rho}{2}\boldsymbol{\pi}^2(\mathbf{x})\right] + \mathcal{O}(T^2)\\ &\equiv -\beta\mathcal{H}_0 - \beta U\end{aligned}$$

Here $(\boldsymbol{\pi}\cdot\nabla\boldsymbol{\pi})^2 = (\pi_i\partial_k\pi_i)(\pi_j\partial_k\pi_j)$ in the Einstein summation convention.

(f) Thus we have a perturbation βU to the Gaussian Hamiltonian $\beta\mathcal{H}_0$. This βU consists of two terms: the $(\boldsymbol{\pi}\cdot\nabla\boldsymbol{\pi})^2$ and the $\boldsymbol{\pi}^2$ terms, which we will write as $\beta U = \beta U_1 + \beta U_2$. Show that the Fourier transforms of βU_1 and βU_2 are:

$$\begin{aligned}\beta U_1 &= \frac{K}{2}\int d^d\mathbf{x}\,\left(\boldsymbol{\pi}(\mathbf{x})\cdot\nabla\boldsymbol{\pi}(\mathbf{x})\right)^2\\ &= -\frac{K}{2}\int_0^\Lambda \frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2\, d^d\mathbf{q}_3}{(2\pi)^{3d}}(\mathbf{q}_1\cdot\mathbf{q}_3)\pi_i(\mathbf{q}_1)\pi_i(\mathbf{q}_2)\pi_j(\mathbf{q}_3)\pi_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\\ \beta U_2 &= -\frac{\rho}{2}\int d^d\mathbf{x}\,\boldsymbol{\pi}^2(\mathbf{x}) = -\frac{\rho}{2}\int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}\pi_i(\mathbf{q})\pi_i(-\mathbf{q})\end{aligned}$$

. . .

We construct the RG transformation just like in class. Define slow and fast modes:

$$\boldsymbol{\pi}(\mathbf{q}) = \begin{cases} \boldsymbol{\pi}_<(\mathbf{q}) & 0 < \mathbf{q} < \Lambda/b \\ \boldsymbol{\pi}_>(\mathbf{q}) & \Lambda/b < \mathbf{q} < \Lambda \end{cases}$$

We break $\beta\mathcal{H}_0$ into slow and fast mode parts, $\beta\mathcal{H}_0 = \beta\mathcal{H}_{0<} + \beta\mathcal{H}_{0>}$, and then integrate the full partition function Z with respect to the fast modes, giving:

$$Z = Z_{0>} \int \mathcal{D}\pi_< \, \exp\left(-\beta\mathcal{H}_{0<} + \ln\langle e^{-\beta U}\rangle_{0>}\right)$$

We thus have some effective slow-mode Hamiltonian $-\beta\tilde{\mathcal{H}}$, which we can write as:

$$\begin{aligned}
-\beta\tilde{\mathcal{H}} &= -\beta\mathcal{H}_{0<} + \ln\langle e^{-\beta U}\rangle_{0>} \\
&= -\frac{\tilde{K}}{2}\int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}\, q^2 \pi_{i<}(\mathbf{q})\pi_{i<}(-\mathbf{q}) \\
&\quad + \frac{\tilde{L}}{2}\int_0^{\Lambda/b} \frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2\, d^d\mathbf{q}_3}{(2\pi)^{3d}}\, (\mathbf{q}_1\cdot\mathbf{q}_3)\pi_{i<}(\mathbf{q}_1)\pi_{i<}(\mathbf{q}_2)\pi_{j<}(\mathbf{q}_3)\pi_{j<}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \\
&\quad + \frac{\tilde{\rho}}{2}\int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}\, \pi_{i<}(\mathbf{q})\pi_{i<}(-\mathbf{q}) + \cdots
\end{aligned}$$

for some new parameters $\tilde{K}$, $\tilde{L}$, $\tilde{\rho}$, etc. Note something interesting here: in the effective Hamiltonian $-\beta\mathcal{H}$ we give different coefficients $\tilde{K}$ and $\tilde{L}$ to the $(\nabla\boldsymbol{\pi})^2$ term and the $(\boldsymbol{\pi}\cdot\nabla\boldsymbol{\pi})^2$ term. But in the Taylor expansion of the full Hamiltonian in part (e), these two terms had the *same* coefficient K. These terms having identical coefficients is a direct consequence of the rotational symmetry of the Hamiltonian. Since an RG transformation always preserves symmetries, the effective Hamiltonian should also have rotational symmetry: this means that $K' = L'$ for an exact RG transformation (the primed variables are the coefficients after making the rescaling $\mathbf{q}' = b\mathbf{q}$ and replacing $\boldsymbol{\pi}$ by $\boldsymbol{\pi}'$). But we cannot apply RG exactly on this system; we have to do it order by order in the cumulant expansion. Thus at first order in RG, we might have $K' \neq L'$. This is an artifact of our approximation. If we carried out the RG to all orders, we should find $K' = L'$. We can make a similar argument for the constant ρ, which describes the density of spins in our system, $\rho = N/V$. Under an exact RG transformation, ρ' should equal ρ. We show this as follows: in the renormalized system the number of degrees of freedom is $N' = b^{-d}N$. But we also rescale our units so that $\mathbf{x}' = \mathbf{x}/b$, implying a new volume $V' = b^{-d}V$. Thus $\rho' = \rho$. Practically, this makes our life much easier: the only variable whose flow we care about under RG is K.

(g) Let us now do RG to first order, expanding

$$\ln\langle e^{-\beta U}\rangle_{0>} \approx -\langle\beta U\rangle_{0>} + \cdots = -\langle\beta U_1\rangle_{0>} - \langle\beta U_2\rangle_{0>} + \cdots$$

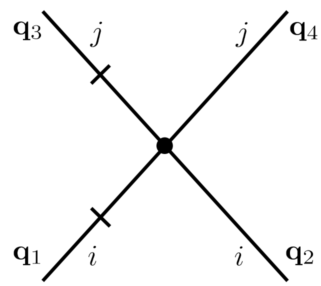
Start with the easy part, $-\langle\beta U_2\rangle_{0>}$. Argue that βU_2 can be broken up into a slow mode and a fast mode piece. Thus show that:

$$-\langle\beta U_2\rangle_{0>} = \frac{\rho}{2}\int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}\, \pi_{i<}(\mathbf{q})\pi_{i<}(-\mathbf{q}) + \text{constant}$$

This means that $-\langle \beta U_2 \rangle_{0>}$ contributes to the $\tilde{\rho}$ term in $-\beta\tilde{\mathcal{H}}$. Now consider the harder part,

$$-\langle \beta U_1 \rangle_{0>} = \frac{K}{2} \int_0^\Lambda \frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2\, d^d\mathbf{q}_3}{(2\pi)^{3d}} (\mathbf{q}_1 \cdot \mathbf{q}_3) \langle \pi_i(\mathbf{q}_1)\pi_i(\mathbf{q}_2)\pi_j(\mathbf{q}_3)\pi_j(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \rangle_{0>}$$

Clearly this term mixes fast and slow modes, so we need to use Wick's theorem and the diagram techniques discussed in class. The vertex diagram corresponding to this term looks like:



Note that the $\mathbf{q}_1$ and $\mathbf{q}_3$ legs are slashed: they are special and need to be distinguished, since we have a factor $(\mathbf{q}_1 \cdot \mathbf{q}_3)$ inside the integral in $-\beta U_1$. Find all the nonzero diagrams that can be constructed from this vertex, and their multiplicities. Ignore diagrams that add constant terms to the effective Hamiltonian. Evaluate the diagrams (the integrals are easy!). Putting the contributions from $-\beta\mathcal{H}_{0<}$, $-\langle \beta U_1 \rangle_{0>}$, and $-\langle \beta U_2 \rangle_{0>}$ together, show that:

$$\tilde{K} = K + \frac{S_d \Lambda^{d-2}(1 - b^{2-d})}{(2\pi)^d (d-2)}, \quad \tilde{L} = K, \quad \tilde{\rho} = b^{-d}\rho$$

Hint: Remember that if we have an odd function $f(\mathbf{q})$, where $f(-\mathbf{q}) = -f(\mathbf{q})$, then $\int d^d\mathbf{q}\, f(\mathbf{q}) = 0$. Also, note that we can write $\rho = N/V$ in an alternative form, using the fact that there are N modes $\mathbf{q}$ in the Brillouin zone:

$$\rho = \frac{N}{V} = \frac{1}{V} \sum_{\mathbf{q} \in \text{B.Z.}} = \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} = \frac{S_d \Lambda^d}{d(2\pi)^d}$$

(h) Perform the rescaling and renormalizing steps: define new momenta $\mathbf{q}' \equiv b\mathbf{q}$ and new Fourier modes $\boldsymbol{\pi}'(\mathbf{q}') \equiv \zeta^{-1}\boldsymbol{\pi}_<(\mathbf{q})$. Show that the RG equations are:

$$K' = \zeta^2 b^{-d-2}\tilde{K}, \quad L' = \zeta^4 b^{-3d-2}\tilde{L}, \quad \rho' = \zeta^2 b^{-d}\tilde{\rho}$$

(i) How do we choose the parameter ζ? The renormalized Hamiltonian should satisfy the same constraints as the original: we fix ζ so that the renormalized spin in position space, $\mathbf{s}'(\mathbf{x}') = \{\pi'_1(\mathbf{x}'), \ldots, \pi'_{n-1}(\mathbf{x}'), \sigma'(\mathbf{x}')\}$, has unit length. But what are the transformation equations for $\boldsymbol{\pi}'(\mathbf{x}')$ and $\sigma'(\mathbf{x}')$ in position space? To find these, let us start with the momentum space definition: $\pi'_i(\mathbf{q}') = \zeta^{-1}\pi_{i<}(\mathbf{q})$. Plug this into the Fourier expansion for $\pi'_i(\mathbf{x}')$,

$$\pi'_i(\mathbf{x}') = \int_0^\Lambda \frac{d^d\mathbf{q}'}{(2\pi)^d}\, e^{i\mathbf{q}'\cdot\mathbf{x}'}\pi'_i(\mathbf{q}')$$

and show that we can write $\pi'_i(\mathbf{x}') = b^d\zeta^{-1}\pi_{i<}(\mathbf{x})$, where we define:

$$\pi_{i<}(\mathbf{x}) \equiv \int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}\, e^{i\mathbf{q}\cdot\mathbf{x}}\pi_{i<}(\mathbf{q})$$

Alternatively, show that we can rewrite $\pi_{i<}(\mathbf{x})$ as:

$$\pi_{i<}(\mathbf{x}) = \left\langle \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d}\, e^{i\mathbf{q}\cdot\mathbf{x}}\pi_i(\mathbf{q}) \right\rangle_{0>} = \langle \pi_i(\mathbf{x})\rangle_{0>}$$

To get this identity, remember that $\langle \pi_{i<}(\mathbf{q})\rangle_{0>} = \pi_{i<}(\mathbf{q})$ and $\langle \pi_{i>}(\mathbf{q})\rangle_{0>} = 0$, by the properties of averages with respect to the Gaussian $\mathcal{H}_{0>}$. Thus we have:

$$\pi'_i(\mathbf{x}') = b^d\zeta^{-1}\langle \pi_i(\mathbf{x})\rangle_{0>}$$

(j) The system has rotational symmetry, so a scaling equation obeyed along one direction should be obeyed along all other directions. This implies an analogous scaling equation for the nth component of $\mathbf{s}'(\mathbf{x}')$:

$$\sigma'(\mathbf{x}') = b^d\zeta^{-1}\langle \sigma(\mathbf{x})\rangle_{0>}$$

Let us calculate $\langle \sigma(\mathbf{x})\rangle_{0>}$. First, note that $\sigma(\mathbf{x}) = \sqrt{1 - \pi_i(\mathbf{x})\pi_i(\mathbf{x})}$, and Taylor expand this to lowest order in $\pi_i(\mathbf{x})\pi_i(\mathbf{x})$. Then Fourier transform and use the properties of $\langle\cdots\rangle_{0>}$ averages to show that:

$$\langle \sigma(\mathbf{x})\rangle_{0>} = 1 - \frac{1}{2}\pi_{i<}(\mathbf{x})\pi_{i<}(\mathbf{x}) - \frac{(n-1)}{2K}\frac{S_d\Lambda^{d-2}(1-b^{2-d})}{(2\pi)^d(d-2)} + \mathcal{O}(T^2)$$

(k) Finally, combine the results of parts (i) and (j) to show that the constraint

$$\mathbf{s}'(\mathbf{x}')^2 = \pi'_i(\mathbf{x}')\pi'_i(\mathbf{x}') + \sigma'(\mathbf{x}')^2 = 1$$

implies that:

$$\zeta = b^d\left(1 - \frac{(n-1)}{2K}\frac{S_d\Lambda^{d-2}(1-b^{2-d})}{(2\pi)^d(d-2)} + \mathcal{O}(T^2)\right)$$

(l) If you are reading this and have successfully completed parts (a)-(k), don't despair: the worst is over. Plug in the result for ζ from part (k) into the RG equation for K' from part (h). Show that:

$$K' = b^{d-2}\left(K - (n-2)\frac{S_d\Lambda^{d-2}(1-b^{2-d})}{(2\pi)^d(d-2)} + \mathcal{O}(T^2)\right)$$

For an infinitesimal rescaling $b = e^{\delta l} \approx 1 + \delta l$, show that the RG equation becomes $K' = K + (dK/dl)\delta l + \cdots$, where:

$$\frac{dK}{dl} = (d-2)K - \frac{(n-2)S_d\Lambda^{d-2}}{(2\pi)^d}$$

This implies an RG equation for the temperature $T = K^{-1}$:

$$\frac{dT}{dl} = -\frac{1}{K^2}\frac{dK}{dl} = (2-d)T + \frac{(n-2)S_d\Lambda^{d-2}}{(2\pi)^d}T^2$$

(m) Consider the case $d \leq 2$. Using the fixed point equation $\frac{dT}{dl}\Big|_{T^*} = 0$, demonstrate that the only low temperature fixed point is $T^* = 0$. For small temperatures T near $T^* = 0$, show that $T' \approx b^{y_T}T$, where the thermal eigenvalue exponent $y_T = 2 - d \geq 0$. Thus small T flow to higher values, and eventually to the disordered sink at $T^* = \infty$. There is no order in the system except at $T = 0$. Draw a flow diagram for the T axis, showing the fixed point and flow behavior.

(n) Consider the case $d > 2$, $n > 2$. Show that in addition to the zero temperature fixed point, there is now another fixed point at:

$$T^* = \frac{(d-2)(2\pi)^d}{(n-2)S_d\Lambda^{d-2}}$$

Calculate the thermal eigenvalue y_T at T^* and show that $y_T = d - 2$. Draw a flow diagram for the T axis, showing the fixed point and flow behavior. This T^* is the critical temperature separating the disordered phase at $T > T^*$ from the ordered phase at $T < T^*$. Note that $T^* \to 0$ as d approaches 2 from above (looking at d as a continuous variable that can take non-integer values). If we define $\epsilon \equiv d - 2$, then $T^* \sim \mathcal{O}(\epsilon)$. Since our whole RG approach was based on a low temperature expansion, we expect our results to be closest to reality when ϵ is small.

(o) Notice that for $n = 2$, $d = 2$, the T^* equation from part (n) is indeterminate. This reflects the fact that the $n = 2$, $d = 2$ system (known as the XY model) is special: it turns out that long-range order is still destroyed at all T, but we need to include other effects besides Goldstone modes (i.e. vortices), which are not described by the nonlinear σ-model. What about $d = 2$, $n > 2$? According to part (m), in this case there is a fixed point at $T^* = 0$, and $y_T = 0$. Consider the critical exponent $\nu = 1/y_T$, describing the behavior of the correlation length $\xi \sim T^\nu$ as $T \to 0$. When $y_T = 0$, we have $\nu = \infty$. What does this mean physically? To find out, look at the RG equation for dT/dl at $d = 2$, $n > 2$. Show that it is possible to integrate this equation directly, obtaining $T(l)$ as a function of l, where $b = e^l$:

$$\frac{1}{T(l)} = \frac{1}{T(0)} - Cl$$

Here C is a positive constant and $T(0) = T$ is the temperature of the original system. For any $T(0) > 0$, $T(l)$ increases with increasing l, flowing to larger temperature values. The correlation behaves like $\xi' = b^{-1}\xi$ under the RG transformation (this is a simple consequence of the length rescaling $\mathbf{x}' = \mathbf{x}/b$), so we can write $\xi(l) = e^{-l}\xi(0)$, where $\xi(0) = \xi$ is the correlation length in the original system. As l increases, $\xi(l)$ decreases. The smallest that $\xi(l)$ can be is just the lattice spacing a (we wrote it as ℓ earlier, but here let us denote it by a so as not to confuse l and ℓ). The equality $\xi(l) = a$ must occur at infinite temperature $T(l) = \infty$ when the renormalized system is totally disordered. Using this fact, show that:

$$\xi = ae^{1/CT}$$

Thus as $T \to 0$, the correlation length for $d = 2$, $n > 2$ blows up like an exponential, faster than any power law.