RG Methods in Statistical Field Theory: Problem Set 4 Solution

In class we discussed the phase transition in superconductors, described by a complex order parameter $\psi(\mathbf{x})$ where $|\psi(\mathbf{x})|^2 = n_s$, the density of superconducting electrons. In this problem set we will look at this transition in more detail, deriving one of the most dramatic consequences of superconductivity: the photons inside a superconductor actually gain an effective mass. This is known as the *Anderson-Higgs mechanism*, and occurs when a local gauge symmetry is spontaneously broken. This idea occupies a central place in particle physics, as the mechanism by which all elementary particles acquire mass. We will see that a direct consequence of massive photons is the *Meissner effect*: the expulsion of magnetic fields from the interior of a superconductor.

Our starting point is the Landau-Ginzburg Hamiltonian for a three-dimensional superconductor in the presence an electromagnetic vector potential $\mathbf{A}(\mathbf{x})$ associated with the magnetic field $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$:

$$\mathcal{H} = \int d^3\mathbf{x} \left[\frac{r}{2}|\psi(\mathbf{x})|^2 + u|\psi(\mathbf{x})|^4 + \frac{K}{2} D_\alpha \psi(\mathbf{x}) D_\alpha^* \psi^*(\mathbf{x}) + \frac{1}{8\pi}(\nabla \times \mathbf{A}(\mathbf{x}))^2 \right]$$

Here $r = a(T - T_c)$, $u > 0$, and the operator D_α is defined as:

$$D_\alpha \equiv \partial_\alpha - ieA_\alpha$$

where $\partial_\alpha \equiv \partial/\partial x_\alpha$ is the spatial derivative along the αth direction, and A_α is the αth component of $\mathbf{A}$. The Einstein summation convention is used in the $D_\alpha\psi(\mathbf{x})D_\alpha^*\psi^*(\mathbf{x})$ term, and will be assumed throughout the rest of the problem set. The constants K and e are related to the microscopic parameters describing the Cooper pairs: $K = \hbar^2/m_c$, $e = e_c/\hbar c$, where m_c and e_c are the mass and charge of a Cooper pair. The last term in the Hamiltonian is just the magnetic field energy density $\frac{1}{8\pi}|\mathbf{B}(\mathbf{x})|^2$.

(a) The $\frac{K}{2}D_\alpha\psi D_\alpha^*\psi^*$ term in the Hamiltonian above has the same role as the $\frac{c}{2}(\nabla m)^2 = \frac{c}{2}\partial_\alpha m \partial_\alpha m$ term in the normal Landau-Ginzburg Hamiltonian we are familiar with from class. But why do we use the operator D_α instead of just ∂_α? This has to do with the fact that our Hamiltonian must be invariant under gauge transformations. We know that in ordinary electromagnetism the physics of our system is unchanged if we replace the vector potential $\mathbf{A}(\mathbf{x})$ with $\mathbf{A}(\mathbf{x}) + \nabla\Lambda(\mathbf{x})$, where $\Lambda(\mathbf{x})$ is an arbitrary function. For example the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ remains unchanged, because $\nabla \times \nabla\Lambda(\mathbf{x}) = 0$. In quantum mechanics a gauge transformation involves changing both the vector potential and the wavefunction:

$$A_\alpha(\mathbf{x}) \mapsto A_\alpha(\mathbf{x}) + \frac{1}{e}\partial_\alpha\theta(\mathbf{x}), \qquad \psi(\mathbf{x}) \mapsto \psi(\mathbf{x})e^{i\theta(\mathbf{x})}$$

where we have written $\Lambda(\mathbf{x})$ as $\frac{1}{e}\theta(\mathbf{x})$. Show that the Landau-Ginzburg Hamiltonian for a superconductor is invariant under this gauge transformation. We call the operator D_α the gauge-invariant derivative.

<u>**Answer:**</u> Under the gauge transformation the first two terms in the Hamiltonian become:

$$\frac{r}{2}|\psi(\mathbf{x})e^{i\theta(\mathbf{x})}|^2 + u|\psi(\mathbf{x})e^{i\theta(\mathbf{x})}|^4 = \frac{r}{2}|\psi(\mathbf{x})|^2 + u|\psi(\mathbf{x})|^4$$

so these terms are unchanged. The third term becomes:

$$\begin{aligned}
&\frac{K}{2}\left[(\partial_\alpha - ieA_\alpha(\mathbf{x}) - i\partial_\alpha\theta(\mathbf{x}))(\psi(\mathbf{x})e^{i\theta(\mathbf{x})})\right]\left[(\partial_\alpha + ieA_\alpha(\mathbf{x}) + i\partial_\alpha\theta(\mathbf{x}))(\psi^*(\mathbf{x})e^{-i\theta(\mathbf{x})})\right] \\
&= \frac{K}{2}\left[e^{i\theta(\mathbf{x})}(\partial_\alpha + i\partial_\alpha\theta(\mathbf{x}) - ieA_\alpha(\mathbf{x}) - i\partial_\alpha\theta(\mathbf{x}))\psi(\mathbf{x})\right] \\
&\qquad \cdot \left[e^{-i\theta(\mathbf{x})}(\partial_\alpha - i\partial_\alpha\theta(\mathbf{x}) + ieA_\alpha(\mathbf{x}) + i\partial_\alpha\theta(\mathbf{x}))\psi^*(\mathbf{x})\right] \\
&= \frac{K}{2}\left[(\partial_\alpha - ieA_\alpha(\mathbf{x}))\psi(\mathbf{x})\right]\left[(\partial_\alpha + ieA_\alpha(\mathbf{x}))\psi^*(\mathbf{x})\right] \\
&= \frac{K}{2}D_\alpha\psi(\mathbf{x})D^*_\alpha\psi^*(\mathbf{x})
\end{aligned}$$

thus this term is also unchanged. Finally the last term in the Hamiltonian becomes:

$$\begin{aligned}
\frac{1}{8\pi}\left(\nabla \times \left[\mathbf{A}(\mathbf{x}) + \frac{1}{e}\nabla\theta(\mathbf{x})\right]\right)^2 &= \frac{1}{8\pi}\left(\nabla \times \mathbf{A}(\mathbf{x}) + \frac{1}{e}\nabla \times \nabla\theta(\mathbf{x})\right)^2 \\
&= \frac{1}{8\pi}\left(\nabla \times \mathbf{A}(\mathbf{x})\right)^2
\end{aligned}$$

and is also unchanged. We conclude that the Hamiltonian is invariant under the gauge transformation.

(b) Let us solve the problem using a mean-field approximation, where we write the partition function as $Z \approx \exp(-\beta\mathcal{H}[\psi_{\text{sad}}(\mathbf{x}), \mathbf{A}_{\text{sad}}(\mathbf{x})])$, where $\psi_{\text{sad}}(\mathbf{x})$ and $\mathbf{A}_{\text{sad}}(\mathbf{x})$ minimize the Hamiltonian $\mathcal{H}$, satisfying the saddle-point equations:

$$\frac{\delta\mathcal{H}}{\delta\psi^*(\mathbf{x})} = 0\,, \qquad \frac{\delta\mathcal{H}}{\delta A_\alpha(\mathbf{x})} = 0$$

In calculating these functional derivatives, treat $\psi(\mathbf{x})$ and $\psi^*(\mathbf{x})$ as independent functions. (There is a third saddle-point equation, the derivative of $\mathcal{H}$ with respect to $\psi(\mathbf{x})$, but it gives no new information.) Show that the saddle point equations can be written as:

$$\begin{aligned}
\frac{r}{2}\psi(\mathbf{x}) + 2u\psi(\mathbf{x})|\psi(\mathbf{x})|^2 - \frac{K}{2}D_\alpha D_\alpha\psi(\mathbf{x}) = 0, \\
\frac{K}{2}\left(-ie\psi(\mathbf{x})D^*_\alpha\psi^*(\mathbf{x}) + ie\psi^*(\mathbf{x})D_\alpha\psi(\mathbf{x})\right) - \frac{1}{4\pi}\epsilon_{\gamma\sigma\alpha}\epsilon_{\gamma\mu\nu}\partial_\sigma\partial_\mu A_\nu(\mathbf{x}) = 0
\end{aligned}$$

Hint: Write the γth component of the curl $\nabla \times \mathbf{A}$ in the following form:

$$(\nabla \times \mathbf{A})_\gamma = \epsilon_{\gamma\mu\nu}\partial_\mu A_\nu$$

where all repeated indices are summed over, and $\epsilon_{\gamma\mu\nu}$ is the totally antisymmetric tensor, whose components are equal to zero except for $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$.

<u>**Answer:**</u> We can write the Hamiltonian as:

$$\begin{aligned}
\mathcal{H} = \int d^3\mathbf{x}' \Bigg[&\frac{r}{2}\psi(\mathbf{x}')\psi^*(\mathbf{x}') + u\psi(\mathbf{x}')^2\psi^*(\mathbf{x}')^2 \\
&+ \frac{K}{2}\left[(\partial_\gamma - ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\left[(\partial_\gamma + ieA_\gamma(\mathbf{x}'))\psi^*(\mathbf{x}')\right] + \frac{1}{8\pi}\epsilon_{\gamma\sigma\tau}\partial_\sigma A_\tau(\mathbf{x}')\epsilon_{\gamma\mu\nu}\partial_\mu A_\nu(\mathbf{x}')\Bigg]
\end{aligned}$$

To find the first saddle point equation, let us apply the functional derivative with respect to $\psi^*(\mathbf{x})$ to each part of the Hamiltonian:

$$
\begin{aligned}
&\frac{\delta}{\delta\psi^*(\mathbf{x})}\int d^3\mathbf{x}'\left[\frac{r}{2}\psi(\mathbf{x}')\psi^*(\mathbf{x}')+u\psi(\mathbf{x}')^2\psi^*(\mathbf{x}')^2\right] \\
&\quad=\int d^3\mathbf{x}'\left[\frac{r}{2}\psi(\mathbf{x}')\frac{\delta\psi^*(\mathbf{x}')}{\delta\psi^*(\mathbf{x})}+2u\psi(\mathbf{x}')^2\psi^*(\mathbf{x}')\frac{\delta\psi^*(\mathbf{x}')}{\delta\psi^*(\mathbf{x})}\right] \\
&\quad=\int d^3\mathbf{x}'\left[\frac{r}{2}\psi(\mathbf{x}')\delta^{(3)}(\mathbf{x}'-\mathbf{x})+2u\psi(\mathbf{x}')^2\psi^*(\mathbf{x}')\delta^{(3)}(\mathbf{x}'-\mathbf{x})\right] \\
&\quad=\frac{r}{2}\psi(\mathbf{x})+2u\psi(\mathbf{x})^2\psi^*(\mathbf{x}) \\
&\frac{\delta}{\delta\psi^*(\mathbf{x})}\int d^3\mathbf{x}'\frac{K}{2}\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\left[(\partial_\gamma+ieA_\gamma(\mathbf{x}'))\psi^*(\mathbf{x}')\right] \\
&\quad=\int d^3\mathbf{x}'\frac{K}{2}\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\left[(\partial_\gamma+ieA_\gamma(\mathbf{x}'))\frac{\delta\psi^*(\mathbf{x}')}{\delta\psi^*(\mathbf{x})}\right] \\
&\quad=\int d^3\mathbf{x}'\frac{K}{2}\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\left[(\partial_\gamma+ieA_\gamma(\mathbf{x}'))\delta^{(3)}(\mathbf{x}'-\mathbf{x})\right] \\
&\quad=\int d^3\mathbf{x}'\frac{K}{2}\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\partial_\gamma\delta^{(3)}(\mathbf{x}'-\mathbf{x}) \\
&\qquad+\int d^3\mathbf{x}'\frac{K}{2}\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]ieA_\gamma(\mathbf{x}')\delta^{(3)}(\mathbf{x}'-\mathbf{x}) \\
&\quad=-\int d^3\mathbf{x}'\frac{K}{2}\partial_\gamma\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\delta^{(3)}(\mathbf{x}'-\mathbf{x}) \\
&\qquad+\frac{K}{2}\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}))\psi(\mathbf{x})\right]ieA_\gamma(\mathbf{x}) \\
&\quad=-\frac{K}{2}\partial_\gamma\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}))\psi(\mathbf{x})\right]+\frac{K}{2}\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}))\psi(\mathbf{x})\right]ieA_\gamma(\mathbf{x}) \\
&\quad=-\frac{K}{2}\left[\partial_\gamma-ieA_\gamma(\mathbf{x})\right]\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}))\psi(\mathbf{x})\right] \\
&\quad=-\frac{K}{2}D_\gamma D_\gamma\psi(\mathbf{x}) \\
&\frac{\delta}{\delta\psi^*(\mathbf{x})}\int d^3\mathbf{x}'\frac{1}{8\pi}\epsilon_{\gamma\sigma\tau}\partial_\sigma A_\tau(\mathbf{x}')\epsilon_{\gamma\mu\nu}\partial_\mu A_\nu(\mathbf{x}')=0
\end{aligned}
$$

Putting everything together we find:

$$
\frac{r}{2}\psi(\mathbf{x})+2u\psi(\mathbf{x})|\psi(\mathbf{x})|^2-\frac{K}{2}D_\gamma D_\gamma\psi(\mathbf{x})=0
$$

To find the second saddle point equation, let us apply the functional derivative with respect to $A_\alpha(\mathbf{x})$ to each part of the Hamiltonian:

$$
\begin{aligned}
&\frac{\delta}{\delta A_\alpha(\mathbf{x})}\int d^3\mathbf{x}'\left[\frac{r}{2}\psi(\mathbf{x}')\psi^*(\mathbf{x}')+u\psi(\mathbf{x}')^2\psi^*(\mathbf{x}')^2\right]=0 \\
&\frac{\delta}{\delta A_\alpha(\mathbf{x})}\int d^3\mathbf{x}'\frac{K}{2}\left[(\partial_\gamma-ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\left[(\partial_\gamma+ieA_\gamma(\mathbf{x}'))\psi^*(\mathbf{x}')\right] \\
&\quad=-\frac{ieK}{2}\int d^3\mathbf{x}'\frac{\delta A_\gamma(\mathbf{x}')}{\delta A_\alpha(\mathbf{x})}\psi(\mathbf{x}')\left[(\partial_\gamma+ieA_\gamma(\mathbf{x}'))\psi^*(\mathbf{x}')\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{ieK}{2}\int d^3\mathbf{x}'\left[(\partial_\gamma - ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\frac{\delta A_\gamma(\mathbf{x}')}{\delta A_\alpha(\mathbf{x})}\psi^*(\mathbf{x}')\\
&= -\frac{ieK}{2}\int d^3\mathbf{x}'\delta_{\alpha\gamma}\delta^{(3)}(\mathbf{x}'-\mathbf{x})\psi(\mathbf{x}')\left[(\partial_\gamma + ieA_\gamma(\mathbf{x}'))\psi^*(\mathbf{x}')\right]\\
&\quad+\frac{ieK}{2}\int d^3\mathbf{x}'\left[(\partial_\gamma - ieA_\gamma(\mathbf{x}'))\psi(\mathbf{x}')\right]\delta_{\alpha\gamma}\delta^{(3)}(\mathbf{x}'-\mathbf{x})\psi^*(\mathbf{x}')\\
&= -\frac{ieK}{2}\psi(\mathbf{x})\left[(\partial_\alpha + ieA_\alpha(\mathbf{x}))\psi^*(\mathbf{x})\right] + \frac{ieK}{2}\left[(\partial_\alpha - ieA_\alpha(\mathbf{x}))\psi(\mathbf{x})\right]\psi^*(\mathbf{x})\\
&= \frac{K}{2}\left(-ie\psi(\mathbf{x})D_\alpha^*\psi^*(\mathbf{x}) + ie\psi^*(\mathbf{x})D_\alpha\psi(\mathbf{x})\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\delta}{\delta A_\alpha(\mathbf{x})}\int d^3\mathbf{x}'\frac{1}{8\pi}\epsilon_{\gamma\sigma\tau}\partial_\sigma A_\tau(\mathbf{x}')\epsilon_{\gamma\mu\nu}\partial_\mu A_\nu(\mathbf{x}')\\
&= \frac{1}{8\pi}\int d^3\mathbf{x}'\epsilon_{\gamma\sigma\tau}\partial_\sigma\frac{\delta A_\tau(\mathbf{x}')}{\delta A_\alpha(\mathbf{x})}\epsilon_{\gamma\mu\nu}\partial_\mu A_\nu(\mathbf{x}') + \frac{1}{8\pi}\int d^3\mathbf{x}'\epsilon_{\gamma\sigma\tau}\partial_\sigma A_\tau(\mathbf{x}')\epsilon_{\gamma\mu\nu}\partial_\mu\frac{\delta A_\nu(\mathbf{x}')}{\delta A_\alpha(\mathbf{x})}\\
&= \frac{1}{8\pi}\int d^3\mathbf{x}'\epsilon_{\gamma\sigma\tau}\delta_{\tau\alpha}\partial_\sigma\delta^{(3)}(\mathbf{x}'-\mathbf{x})\epsilon_{\gamma\mu\nu}\partial_\mu A_\nu(\mathbf{x}')\\
&\quad+\frac{1}{8\pi}\int d^3\mathbf{x}'\epsilon_{\gamma\sigma\tau}\partial_\sigma A_\tau(\mathbf{x}')\epsilon_{\gamma\mu\nu}\delta_{\nu\alpha}\partial_\mu\delta^{(3)}(\mathbf{x}'-\mathbf{x})\\
&= -\frac{1}{8\pi}\int d^3\mathbf{x}'\epsilon_{\gamma\sigma\alpha}\delta^{(3)}(\mathbf{x}'-\mathbf{x})\epsilon_{\gamma\mu\nu}\partial_\sigma\partial_\mu A_\nu(\mathbf{x}')\\
&\quad-\frac{1}{8\pi}\int d^3\mathbf{x}'\epsilon_{\gamma\sigma\tau}\partial_\mu\partial_\sigma A_\tau(\mathbf{x}')\epsilon_{\gamma\mu\alpha}\delta^{(3)}(\mathbf{x}'-\mathbf{x})\\
&= -\frac{1}{8\pi}\epsilon_{\gamma\sigma\alpha}\epsilon_{\gamma\mu\nu}\partial_\sigma\partial_\mu A_\nu(\mathbf{x}) - \frac{1}{8\pi}\epsilon_{\gamma\mu\alpha}\epsilon_{\gamma\sigma\tau}\partial_\mu\partial_\sigma A_\tau(\mathbf{x})\\
&= -\frac{1}{4\pi}\epsilon_{\gamma\sigma\alpha}\epsilon_{\gamma\mu\nu}\partial_\sigma\partial_\mu A_\nu(\mathbf{x})
\end{aligned}
$$

Putting everything together we find:

$$
\frac{K}{2}\left(-ie\psi(\mathbf{x})D_\alpha^*\psi^*(\mathbf{x}) + ie\psi^*(\mathbf{x})D_\alpha\psi(\mathbf{x})\right) - \frac{1}{4\pi}\epsilon_{\gamma\sigma\alpha}\epsilon_{\gamma\mu\nu}\partial_\sigma\partial_\mu A_\nu(\mathbf{x}) = 0
$$

(c) Show that there is a saddle point solution of the form $\psi(\mathbf{x}) = \psi_0$ and $\mathbf{A}(\mathbf{x}) = 0$, where ψ_0 is a complex number independent of $\mathbf{x}$. Find ψ_0 for $T > T_c$ and $T < T_c$. This tells you the behavior of $n_s = |\psi(\mathbf{x})|^2 = |\psi_0|^2$. Note that $\psi_0 = |\psi_0|e^{i\theta_0}$ has a fixed phase factor θ_0 for all $\mathbf{x}$ (this is like the spins in a magnetic model choosing a definite direction in the ordered state). Thus the superconducting phase below T_c, where $|\psi_0| > 0$, breaks the gauge symmetry of the Hamiltonian in the same way that the ordered phase in a magnet breaks rotational symmetry. For simplicity, we will assume $\theta_0 = 0$ from now on, so that ψ_0 is real.

Answer: Plugging in the solution $\psi_{\text{sad}}(\mathbf{x}) = \psi_0$, $\mathbf{A}_{\text{sad}}(\mathbf{x}) = 0$, we find that the second saddle point equation is satisfied trivially, and the first saddle point equation becomes:

$$
\frac{r}{2}\psi_0 + 2u\psi_0|\psi_0|^2 = 0
$$

The mean-field free energy $A = -\frac{1}{\beta}\ln Z = \mathcal{H}[\psi_{\text{sad}}(\mathbf{x}), \mathbf{A}_{\text{sad}}(\mathbf{x})]$ is given by:

$$
A = V(\frac{r}{2}|\psi_0|^2 + u|\psi_0|^4)
$$

For $T > T_c$ ($r > 0$), the only solution to the first saddle point equation is $\psi = 0$. For $T < T_c$ ($r < 0$), there are additional solutions for $|\psi_0|^2 = -r/4u$, and these give a lower A than the $\psi_0 = 0$ solution. Hence for $T > T_c$, $n_s = |\psi_0|^2 = 0$, and for $T < T_c$, $n_s = -r/4u = a(T_c - T)/4u$.

(d) Let us find the energy of fluctuations around the mean-field solution:

$$\psi(\mathbf{x}) = \psi_0 e^{i\theta(\mathbf{x})}, \qquad A_\alpha(\mathbf{x}) = a_\alpha(\mathbf{x})$$

where $\theta(\mathbf{x})$ is a small phase and $\mathbf{a}(\mathbf{x})$ a small vector potential that varies with position. Plug these into the Landau-Ginzburg Hamiltonian and keep terms only up to second-order in the $\theta(\mathbf{x})$ and $\mathbf{a}(\mathbf{x})$. Show that the Hamiltonian becomes:

$$\mathcal{H} = \mathcal{H}_0 + \int d^3\mathbf{x}\left[\frac{K}{2}\psi_0^2(\nabla\theta(\mathbf{x}))^2 - Ke\psi_0^2\mathbf{a}(\mathbf{x})\cdot\nabla\theta(\mathbf{x}) + \frac{K}{2}e^2\psi_0^2(\mathbf{a}(\mathbf{x}))^2 + \frac{1}{8\pi}(\nabla\times\mathbf{a})^2\right]$$

where $\mathcal{H}_0$ is the mean-field Hamiltonian.

<u>**Answer:**</u> Plugging the expressions for $\psi(\mathbf{x})$ and $A_\alpha(\mathbf{x})$ into the Hamiltonian:

$$\begin{aligned}
\mathcal{H} &= \int d^3\mathbf{x}\Bigg[\frac{r}{2}\psi_0^2 + u\psi_0^4 + \frac{K}{2}\left[(\partial_\alpha - iea_\alpha(\mathbf{x}))\psi_0 e^{i\theta(\mathbf{x})}\right]\left[(\partial_\alpha + iea_\alpha(\mathbf{x}))\psi_0^* e^{-i\theta(\mathbf{x})}\right] \\
&\qquad + \frac{1}{8\pi}\epsilon_{\alpha\sigma\tau}\partial_\sigma a_\tau(\mathbf{x})\epsilon_{\alpha\mu\nu}\partial_\mu a_\nu(\mathbf{x})\Bigg] \\
&= V\left(\frac{r}{2}\psi_0^2 + u\psi_0^4\right) \\
&\qquad + \int d^3\mathbf{x}\Bigg[\frac{K}{2}\psi_0^2\left[(i\partial_\alpha\theta(\mathbf{x}) - iea_\alpha(\mathbf{x}))e^{i\theta(\mathbf{x})}\right]\left[(-i\partial_\alpha\theta(\mathbf{x}) + iea_\alpha(\mathbf{x}))e^{-i\theta(\mathbf{x})}\right] \\
&\qquad + \frac{1}{8\pi}(\nabla\times\mathbf{a})^2\Bigg] \\
&= \mathcal{H}_0 + \int d^3\mathbf{x}\Bigg[\frac{K}{2}\psi_0^2\partial_\alpha\theta(\mathbf{x})\partial_\alpha\theta(\mathbf{x}) - Ke\psi_0^2 a_\alpha(\mathbf{x})\partial_\alpha\theta(\mathbf{x}) + \frac{K}{2}e^2\psi_0^2 a_\alpha(\mathbf{x})a_\alpha(\mathbf{x}) \\
&\qquad + \frac{1}{8\pi}(\nabla\times\mathbf{a})^2\Bigg] \\
&= \mathcal{H}_0 + \int d^3\mathbf{x}\left[\frac{K}{2}\psi_0^2(\nabla\theta(\mathbf{x}))^2 - Ke\psi_0^2\mathbf{a}(\mathbf{x})\cdot\nabla\theta(\mathbf{x}) + \frac{K}{2}e^2\psi_0^2(\mathbf{a}(\mathbf{x}))^2 + \frac{1}{8\pi}(\nabla\times\mathbf{a})^2\right]
\end{aligned}$$

(e) Plug in the Fourier expansions:

$$\theta(\mathbf{x}) = \int\frac{d^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\theta(\mathbf{q}), \qquad a_\alpha(\mathbf{x}) = \int\frac{d^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}a_\alpha(\mathbf{q})$$

Show that the Hamiltonian becomes:

$$\begin{aligned}
\mathcal{H} = \mathcal{H}_0 + \int\frac{d^3\mathbf{q}}{(2\pi)^3}\Bigg[&\frac{K\psi_0^2}{2}q^2\theta(\mathbf{q})\theta(-\mathbf{q}) - iK\psi_0^2 e\mathbf{q}\cdot\mathbf{a}(-\mathbf{q})\theta(\mathbf{q}) \\
&+\frac{K}{2}e^2\psi_0^2\mathbf{a}(\mathbf{q})\cdot\mathbf{a}(-\mathbf{q}) + \frac{1}{8\pi}(\mathbf{q}\times\mathbf{a}(\mathbf{q}))\cdot(\mathbf{q}\times\mathbf{a}(-\mathbf{q}))\Bigg]
\end{aligned}$$

Answer: Plugging in the Fourier expansions term by term:

$$
\begin{aligned}
&\int d^3\mathbf{x}\frac{K}{2}\psi_0^2(\nabla\theta(\mathbf{x}))^2 \\
&\quad = \frac{K}{2}\psi_0^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{q}'}{(2\pi)^3}\int d^3\mathbf{x}\,\mathbf{q}\cdot\mathbf{q}'e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}}\theta(\mathbf{q})\theta(\mathbf{q}') \\
&\quad = \frac{K}{2}\psi_0^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{q}'}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{q}')\mathbf{q}\cdot\mathbf{q}'\theta(\mathbf{q})\theta(\mathbf{q}') \\
&\quad = \frac{K}{2}\psi_0^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}q^2\theta(\mathbf{q})\theta(-\mathbf{q}) \\
&-\int d^3\mathbf{x}\,Ke\psi_0^2\mathbf{a}(\mathbf{x})\cdot\nabla\theta(\mathbf{x}) \\
&\quad = -Ke\psi_0^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{q}'}{(2\pi)^3}\int d^3\mathbf{x}\,i\mathbf{a}(\mathbf{q}')\cdot\mathbf{q}\,\theta(\mathbf{q})e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}} \\
&\quad = -iKe\psi_0^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\mathbf{a}(-\mathbf{q})\cdot\mathbf{q}\,\theta(\mathbf{q}) \\
&\int d^3\mathbf{x}\,\frac{K}{2}e^2\psi_0^2(\mathbf{a}(\mathbf{x}))^2 \\
&\quad = \frac{K}{2}e^2\psi_0^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{q}'}{(2\pi)^3}\int d^3\mathbf{x}\,\mathbf{a}(\mathbf{q})\cdot\mathbf{a}(\mathbf{q}')e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}} \\
&\quad = \frac{K}{2}e^2\psi_0^2\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\mathbf{a}(\mathbf{q})\cdot\mathbf{a}(-\mathbf{q}) \\
&\int d^3\mathbf{x}\,\frac{1}{8\pi}(\nabla\times\mathbf{a}(\mathbf{x}))^2 \\
&\quad = \frac{1}{8\pi}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int\frac{d^3\mathbf{q}'}{(2\pi)^3}\int d^3\mathbf{x}\,(i\mathbf{q}\times\mathbf{a}(\mathbf{q}))\cdot(i\mathbf{q}'\times\mathbf{a}(\mathbf{q}'))e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}} \\
&\quad = \frac{1}{8\pi}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,(\mathbf{q}\times\mathbf{a}(\mathbf{q}))\cdot(\mathbf{q}\times\mathbf{a}(-\mathbf{q}))
\end{aligned}
$$

Putting these all together gives us the desired form for the Hamiltonian.

(f) The partition function can be written as a functional integral over all possible fluctuations $\theta(\mathbf{q})$ and $\mathbf{a}(\mathbf{q})$:

$$Z = \int \mathcal{D}\theta(\mathbf{q})\,\mathcal{D}\mathbf{a}(\mathbf{q})\,e^{-\beta\mathcal{H}}$$

Let us do the integration over the phase fluctuations $\theta(\mathbf{q})$ only, which can be carried out using the Gaussian functional integral formulas derived in class. Show that we can write the partition function as:

$$Z \propto \int \mathcal{D}\mathbf{a}(\mathbf{q})e^{-\beta\tilde{\mathcal{H}}}$$

where we have ignored a constant factor in front (since it does not affect the physics we are interested in), and the effective Hamiltonian $\tilde{\mathcal{H}}$ involves only the $\mathbf{a}(\mathbf{q})$ fluctuations:

$$
\begin{aligned}
\tilde{\mathcal{H}} = \int\frac{d^d\mathbf{q}}{(2\pi)^3}\Bigg[&\frac{K\psi_0^2e^2}{2}\left(\mathbf{a}(\mathbf{q})\cdot\mathbf{a}(-\mathbf{q}) - \frac{(\mathbf{q}\cdot\mathbf{a}(\mathbf{q}))(\mathbf{q}\cdot\mathbf{a}(-\mathbf{q}))}{q^2}\right) \\
&+\frac{1}{8\pi}(\mathbf{q}\times\mathbf{a}(\mathbf{q}))\cdot(\mathbf{q}\times\mathbf{a}(-\mathbf{q}))\Bigg]
\end{aligned}
$$

Hint: In class we showed that following formula for a Fourier-transformed Gaussian functional integral:

$$\int \mathcal{D}\phi(\mathbf{q}) \exp\left[-\frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^3} K(\mathbf{q})\phi(\mathbf{q})\phi(-\mathbf{q}) + \int \frac{d^d\mathbf{q}}{(2\pi)^3} h(-\mathbf{q})\phi(\mathbf{q})\right]$$
$$= \sqrt{\frac{(2\pi)^\infty}{\det K}} \exp\left[\frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^3} \frac{h(\mathbf{q})h(-\mathbf{q})}{K(\mathbf{q})}\right]$$

where $h(-\mathbf{q}) = h^*(\mathbf{q})$, $\phi(-\mathbf{q}) = \phi^*(\mathbf{q})$. You do not need to evaluate the square root constant in front, since it just gives a constant factor multiplying Z.

__Answer:__ The part of the Hamiltonian $-\beta\mathcal{H}$ which depends on $\theta(\mathbf{q})$ is:

$$-\beta \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left[\frac{K\psi_0^2}{2} q^2 \theta(\mathbf{q})\theta(-\mathbf{q}) - iK\psi_0^2 e\mathbf{q}\cdot\mathbf{a}(-\mathbf{q})\theta(\mathbf{q})\right]$$

This has a Gaussian form with $K(\mathbf{q}) = \beta K\psi_0^2 q^2$ and $h(-\mathbf{q}) = i\beta K\psi_0^2 e\mathbf{q}\cdot\mathbf{a}(-\mathbf{q})$. Thus integrating over the $\theta(\mathbf{q})$ fluctuations will contribute a factor to the partition function:

$$\exp\left[\frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^3} \frac{h(\mathbf{q})h(-\mathbf{q})}{K(\mathbf{q})}\right] = \exp\left[\frac{\beta K\psi_0^2 e^2}{2} \int \frac{d^d\mathbf{q}}{(2\pi)^3} \frac{(\mathbf{q}\cdot a(\mathbf{q}))(\mathbf{q}\cdot a(-\mathbf{q}))}{q^2}\right]$$

Combining this with the terms in $-\beta\mathcal{H}$ which do not depend on $\theta(\mathbf{q})$, we get the effective Hamiltonian $\tilde{\mathcal{H}}$ quoted above.

(g) To simplify the expression for $\tilde{\mathcal{H}}$ let us decompose the vector $\mathbf{a}(\mathbf{q})$ into components perpendicular and parallel to $\mathbf{q}$ (these are known as the transverse and longitudinal components respectively):

$$\mathbf{a}(\mathbf{q}) = \mathbf{a}^\perp(\mathbf{q}) + \mathbf{a}^\parallel(\mathbf{q})$$

where

$$\mathbf{a}^\perp(\mathbf{q}) \equiv \mathbf{a}(\mathbf{q}) - \frac{\mathbf{q}(\mathbf{q}\cdot\mathbf{a}(\mathbf{q}))}{q^2}, \qquad \mathbf{a}^\parallel(\mathbf{q}) \equiv \frac{\mathbf{q}(\mathbf{q}\cdot\mathbf{a}(\mathbf{q}))}{q^2}$$

Note that $\mathbf{q}\cdot\mathbf{a}^\perp(\mathbf{q}) = 0$ and $\mathbf{q}\times\mathbf{a}^\parallel(\mathbf{q}) = 0$. Show that $\tilde{\mathcal{H}}$ can be written in the form:

$$\tilde{\mathcal{H}} = \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^3} \left(K\psi_0^2 e^2 + \frac{1}{4\pi}q^2\right) \mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q})$$

How can we interpret this Hamiltonian physically? It gives the energy for transverse electromagnetic fluctuations of wavevector $\mathbf{q}$, in other words photons propagating with wavevector $\mathbf{q}$. When $T > T_c$, we are in the normal phase where $n_s = \psi_0^2 = 0$, and the energy for a $\mathbf{q}$ fluctuation is proportional to q^2. When we go to the long wavelength limit, $q \to 0$, the energy becomes arbitrarily small: this corresponds to the photon having zero mass. (Remember the relativistic equation for the energy of a particle is $E^2 = c^2p^2 + m^2c^4$. For a massless photon $m = 0$ and $p = \hbar q$, so the energy goes to zero as $q \to 0$). On the other hand, when $T < T_c$, $n_s = \psi_0^2 > 0$, and we see something very different in the $q \to 0$ limit: the energy of a fluctuation does not go to zero, but is bounded from below by the $K\psi_0^2 e^2$ term. The photon has acquired an effective mass in the superconducting phase.

__Answer:__ Note that since $\mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\parallel(-\mathbf{q}) = \mathbf{a}^\parallel(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q}) = 0$, we can write $\mathbf{a}(\mathbf{q})\cdot\mathbf{a}(-\mathbf{q}) = \mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q}) + \mathbf{a}^\parallel(\mathbf{q})\cdot\mathbf{a}^\parallel(-\mathbf{q})$. Using the definition of $\mathbf{a}^\parallel(\mathbf{q})$ above we can also see that:

$$\mathbf{a}^\parallel(\mathbf{q})\cdot\mathbf{a}^\parallel(-\mathbf{q}) = \frac{(\mathbf{q}\cdot\mathbf{a}(\mathbf{q}))(\mathbf{q}\cdot\mathbf{a}(-\mathbf{q}))}{q^2}$$

Thus the effective Hamiltonian $\tilde{\mathcal{H}}$ can be rewritten as:

$$\begin{aligned}\tilde{\mathcal{H}} &= \int \frac{d^d\mathbf{q}}{(2\pi)^3}\left[\frac{K\psi_0^2e^2}{2}\left(\mathbf{a}(\mathbf{q})\cdot\mathbf{a}(-\mathbf{q}) - \frac{(\mathbf{q}\cdot\mathbf{a}(\mathbf{q}))(\mathbf{q}\cdot\mathbf{a}(-\mathbf{q}))}{q^2}\right)\right.\\ &\quad\left.+\frac{1}{8\pi}(\mathbf{q}\times\mathbf{a}(\mathbf{q}))\cdot(\mathbf{q}\times\mathbf{a}(-\mathbf{q}))\right]\\ &= \int \frac{d^d\mathbf{q}}{(2\pi)^3}\left[\frac{K\psi_0^2e^2}{2}\mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q}) + \frac{1}{8\pi}(\mathbf{q}\times\mathbf{a}^\perp(\mathbf{q}))\cdot(\mathbf{q}\times\mathbf{a}^\perp(-\mathbf{q}))\right]\end{aligned}$$

Using the identity $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$ and the fact that $\mathbf{q}\cdot\mathbf{a}^\perp(\mathbf{q}) = 0$, we can simplify the second term above to get:

$$\begin{aligned}\tilde{\mathcal{H}} &= \int \frac{d^d\mathbf{q}}{(2\pi)^3}\left[\frac{K\psi_0^2e^2}{2}\mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q}) + \frac{1}{8\pi}(\mathbf{q}\cdot\mathbf{q})(\mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q}))\right]\\ &= \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^3}\left(K\psi_0^2e^2 + \frac{1}{4\pi}q^2\right)\mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q})\end{aligned}$$

(h) Let us examine the consequences of massive photons. Look at the Hamiltonian $\tilde{\mathcal{H}}$: it has a nice Gaussian form. What is the correlation function $\langle\mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q})\rangle$? (No complicated calculations are necessary.) Rewrite the correlation function to make it look like:

$$\langle\mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q})\rangle = \frac{C_0}{1+q^2\lambda^2}$$

Find C_0 and λ. Note that λ has units of length. The Fourier transform of this correlation function is $\langle\mathbf{a}^\perp(\mathbf{x})\cdot\mathbf{a}^\perp(\mathbf{x}')\rangle$, and we know from similar examples in earlier lectures that when $\lambda \neq \infty$, the correlation function decays at large distances as:

$$\langle\mathbf{a}^\perp(\mathbf{x})\cdot\mathbf{a}^\perp(\mathbf{x}')\rangle \sim \exp(-|\mathbf{x}-\mathbf{x}'|/\lambda)$$

What this tells us is that correlations between transverse vector potential fluctuations are suppressed exponentially with distance inside a superconductor. For example a magnetic field outside a superconductor will only penetrate significantly into a layer of thickness λ near the surface of the superconductor, and be exponentially small deep in the interior. This is the Meissner effect mentioned earlier. The length λ is known as the *penetration depth.* Plot the behavior of λ as a function of T for $T < T_c$.

__Answer:__ The effective Hamiltonian $\beta\tilde{\mathcal{H}}$ has a Gaussian form with $K(\mathbf{q}) = \beta(K\psi_0^2e^2+q^2/4\pi)$. Thus:

$$\begin{aligned}\langle\mathbf{a}^\perp(\mathbf{q})\cdot\mathbf{a}^\perp(-\mathbf{q})\rangle = \frac{1}{K(\mathbf{q})} &= \frac{1}{\beta(K\psi_0^2e^2+q^2/4\pi)}\\ &= \frac{(\beta K\psi_0^2e^2)^{-1}}{1+q^2(4\pi K\psi_0^2e^2)^{-1}}\end{aligned}$$

Thus:

$$C_0 = \frac{1}{\beta K\psi_0^2e^2} \qquad \lambda = \frac{1}{\sqrt{4\pi K\psi_0^2e^2}}$$

Using the fact that $\psi_0^2 = 0$ for $T > T_c$ and $\psi_0^2 = a(T_c - T)/4u$ for $T < T_c$, we can write:

$$\lambda = \begin{cases} \infty & T > T_c \\ \sqrt{\frac{u}{\pi K e^2 a(T_c - T)}} & T < T_c \end{cases}$$

We plot $\lambda(T)$ below: