RG Methods in Statistical Field Theory: Problem Set 2

due: Friday, October 6, 2006

The ordered phase for the magnetic system we examined in class had a spatially uniform order parameter. Here we will look at a more general situation, with the possibility of a *modulated phase*, where the order parameter varies periodically in space. For simplicity, we consider an $n = 1$, $d = 1$ system along a line of length L, with a Hamiltonian functional given by:

$$\mathcal{H}[m(x)] = \int_0^L dx \left[\frac{r}{2} m^2(x) + u m^4(x) + \frac{c}{2}\left(\frac{\partial m}{\partial x}\right)^2 + \frac{D}{2}\left(\frac{\partial^2 m}{\partial x^2}\right)^2 \right]$$

Here r varies with temperature, and u, D, c are constants. We restrict u, $D > 0$, but we allow c to take on any value. When both $c > 0$ and $D > 0$, any spatial fluctuations in $m(x)$ cost energy, so we expect all phases to be uniform. On the other hand when $c < 0$ and $D > 0$, there is the possibility that the system can lower its free energy by going to a nonuniform phase. We will investigate this possibility by constructing the mean-field phase diagram in terms of r and c. We do this in several steps:

(a) Because we are dealing with the possibility of spatially fluctuating $m(x)$, it is reasonable to rewrite the Hamiltonian in terms of Fourier modes. We define the Fourier transforms:

$$m(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} m_n e^{iq_n x}\,, \qquad m_n = \int_0^L dx\, e^{-iq_n x} m(x)$$

where n is an integer and $q_n = 2\pi n/L$. The orthogonality and completeness properties for the Fourier modes $e^{iq_n x}$ are:

$$\int_0^L dx\, e^{i(q_n - q_{n'})x} = L\delta_{n,n'}\,, \qquad \sum_{n=-\infty}^{\infty} e^{-iq_n x} = L\delta(x)$$

Plug in the expansion for $m(x)$ into the Hamiltonian $\mathcal{H}$ and show that we can write:

$$\mathcal{H} = \frac{1}{2L}\sum_n K_n m_n m_{-n} + \frac{u}{L^3}\sum_{n,n',n''} m_n m_{n'} m_{n''} m_{-n-n'-n''}$$

where $K_n = r + cq_n^2 + Dq_n^4$.

(b) We solve this system using a mean-field approximation, writing the partition function $Z \approx \exp(-\beta\mathcal{H}[m_{\text{sad}}(x)])$, where $m_{\text{sad}}(x)$ minimizes $\mathcal{H}$. In terms of Fourier modes, the saddle point condition can be expressed as the series of coupled equations:

$$\frac{\partial \mathcal{H}}{\partial m_n} = 0 \quad \text{for all } n$$

Show that each of the following cases is a possible solution of the saddle point equations. In each case, also find the range of r and c for which the solution is possible. (Do not worry about the free energy of the solutions yet; we will look at this in the next part.)

Case I: $m_n = 0$ for all n. This case corresponds to an order parameter $m(x) = 0$ at every point.

Case II: $m_n = La_0\delta_{n,0}$ where $a_0 \neq 0$ is a real constant. This case corresponds to a uniform order parameter $m(x) = a_0$.

Case III: $m_n = La_k(\delta_{n,k} + \delta_{n,-k})$ for some positive integer $k \neq 0$ and real constant $a_k \neq 0$. This case corresponds to a spatially varying order parameter $m(x) = 2a_k \cos(2\pi kx/L)$.

c) We would now like to draw a phase diagram in terms of r and c, centered at the point $r = 0$, $c = 0$, and including regions of positive and negative r, and positive and negative c. For any given r and c, the phase at that point is determined by which of the three solutions in part (b) has the smallest free energy $A = -k_BT \ln Z \approx \mathcal{H}$. Case I corresponds to the paramagnetic phase, Case II to the ferromagnetic phase, and Case III to the modulated phase. Note that in Case III, for a given value of c, there will be a single value of k which gives the minimum free energy. You should find this k in terms of c and D, which tells you the wavevector of the order parameter in the modulated phase. When drawing the phase diagram, you should get exact equations for all transition curves in the diagram, and identify which transitions are first-order (discontinuous change of order parameter), and which transitions are second-order (continuous change of order parameter).