RG Methods in Statistical Field Theory: Problem Set 1 Solutions

In lecture, we argued that a field theory can be constructed by looking at a coarse-grained description of a physical system, where the field $\phi(\mathbf{x})$ is a local thermodynamic average. In the problem below, we will show an alternative way of constructing a field theory that can be applied to certain systems, starting directly from the microscopic Hamiltonian. This method is known as a *Hubbard-Stratonovich transformation*.

The physical system we consider is an Ising model on a d-dimensional hypercubic lattice, defined by the Hamiltonian:

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij} s_i s_j \, ,$$

where at each position $\mathbf{x}_i$ in the lattice, we have a spin s_i that can take on one of two values, $s_i = \pm 1$. The lattice spacing is ℓ and there are N sites in total. The interaction between spins is given by the matrix $\mathbf{J}$ with components:

$$J_{ij} = \begin{cases} J & \text{if } i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise}\, , \end{cases}$$

where $J > 0$. For a hypercubic lattice in d dimensions, a site $\mathbf{x}_i$ will have $2d$ nearest neighbors. The partition function for this system is:

$$Z = \sum_{s_1=\pm 1}\sum_{s_2=\pm 1}\cdots\sum_{s_N=\pm 1} e^{-\beta\mathcal{H}} \, .$$

We would like to express this partition function as a field theory. To do this, we first need to prove an identity:

(a) Prove the following result for a general Gaussian integral over the variables ϕ_i, $i = 1,\ldots,N$:

$$\int_{-\infty}^{\infty}\prod_{i=1}^{N} d\phi_i \, \exp\left(-\frac{1}{2}\sum_{i,j}\phi_i A_{ij}\phi_j\right) = \sqrt{\frac{(2\pi)^N}{\det(\mathbf{A})}} \, .$$

Here $\mathbf{A}$ is a real, symmetric $N\times N$ matrix with positive eigenvalues, and we can consider the ϕ_i to be components of an N-dimensional vector $\boldsymbol{\phi}$. *Hint:* To prove this identity, use the fact that the N orthonormal eigenvectors of $\mathbf{A}$ form a basis for our N-dimensional vector space. Let us denote these eigenvectors as $\mathbf{v}_q$, $q = 1,\ldots,N$, and the corresponding eigenvalues λ_q. They satisfy:

$$\mathbf{A}\mathbf{v}_q = \lambda_q \mathbf{v}_q \, , \qquad \mathbf{v}_q\cdot\mathbf{v}_{q'} = \delta_{qq'} \, .$$

We can write the vector $\boldsymbol{\phi}$ in the new basis as a linear combination of $\mathbf{v}_q$: $\boldsymbol{\phi} = \sum_q \tilde{\phi}_q \mathbf{v}_q$, where the $\{\phi_q\}$ are some coefficients. To simplify the integral, do a change of variables from the set $\{\phi_i\}$ to $\{\tilde{\phi}_q\}$. The integral should now look like a product of ordinary Gaussian integrals, which can be solved using the result, $\int_{-\infty}^{\infty} dx\, \exp(-ax^2/2) = \sqrt{2\pi/a}$ for $a > 0$. Keep in mind that the determinant of a matrix is the same in any basis.

Answer: The new variables are defined by $\boldsymbol{\phi} = \sum_q \tilde{\phi}_q \mathbf{v}_q$, or in component notation: $\phi_i = \sum_q \tilde{\phi}_q v_{qi}$, where v_{qi} is the ith component of the eigenvector $\mathbf{v}_q$. Let us plug in this relation to rewrite the term inside the exponential:

$$\begin{aligned}
-\frac{1}{2}\sum_{i,j} \phi_i A_{ij} \phi_j &= -\frac{1}{2}\sum_{i,j} \left(\sum_q \tilde{\phi}_q v_{qi}\right) A_{ij} \left(\sum_{q'} \tilde{\phi}_{q'} v_{q'j}\right) \\
&= -\frac{1}{2}\sum_{q,q'} \tilde{\phi}_q \tilde{\phi}_{q'} \sum_i v_{qi} \sum_j A_{ij} v_{q'j} \\
&= -\frac{1}{2}\sum_{q,q'} \tilde{\phi}_q \tilde{\phi}_{q'} \sum_i \lambda_{q'} v_{qi} v_{q'i} \\
&= -\frac{1}{2}\sum_{q,q'} \tilde{\phi}_q \tilde{\phi}_{q'} \lambda_{q'} \delta_{qq'} \\
&= -\frac{1}{2}\sum_{q} \lambda_q \tilde{\phi}_q^2 .
\end{aligned}$$

In the fourth line we have used the orthonormal property of the eigenvectors, $\mathbf{v}_q \cdot \mathbf{v}_{q'} = \sum_i v_{qi} v_{q'i} = \delta_{qq'}$.

Changing the variables in the integral, we write:

$$\prod_{i=1}^{N} d\phi_i = |\det(\mathbf{J})| \prod_{q=1}^{N} d\tilde{\phi}_q ,$$

where $\mathbf{J}$ is the Jacobian matrix for the transformation, defined as:

$$J_{iq} = \frac{\partial \phi_i}{\partial \tilde{\phi}_q} .$$

From the relation $\phi_i = \sum_q \tilde{\phi}_q v_{qi}$, it is easy to see that $J_{iq} = v_{qi}$, so the qth column of the Jacobian matrix $\mathbf{J}$ is just the eigenvector $\mathbf{v}_q$. Using the orthonormality of the eigenvectors, we can show that $\mathbf{J}$ is an orthogonal matrix, satisfying $\mathbf{J}^T\mathbf{J} = \mathbf{1}$:

$$(\mathbf{J}^T\mathbf{J})_{ij} = \sum_k (J^T)_{ik} J_{kj} = \sum_k v_{ik} v_{jk} = \delta_{ij} .$$

The determinant of a real, orthogonal matrix is equal to 1, because:

$$\det(\mathbf{J}^T\mathbf{J}) = \det(\mathbf{1}) \quad \Rightarrow \quad \det(\mathbf{J}^T)\det(\mathbf{J}) = 1 \quad \Rightarrow \quad \det(\mathbf{J})^2 = 1 \quad \Rightarrow \quad \det(\mathbf{J}) = 1 ,$$

where we have used the fact that $\det(\mathbf{J}^T) = \det(\mathbf{J})$ for any matrix $\mathbf{J}$.

Thus the integral becomes:

$$\int_{-\infty}^{\infty} \prod_{q=1}^{N} d\tilde{\phi}_q \exp\left(-\frac{1}{2}\sum_q \lambda_q \tilde{\phi}_q^2\right) = \prod_q \int_{-\infty}^{\infty} d\tilde{\phi}_q \exp\left(-\frac{1}{2}\lambda_q \tilde{\phi}_q^2\right) = \prod_q \sqrt{\frac{2\pi}{\lambda_q}}$$

In the basis of eigenvectors $\{\mathbf{v}_q\}$ the matrix A is diagonal with the eigenvalues λ_q along the diagonal. Thus $\det(A) = \prod_q \lambda_q$ in this basis. But the determinant is basis-independent, so this result is true in any basis. Finally we get the answer:

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \, \exp\left(-\frac{1}{2}\sum_{i,j} \phi_i A_{ij} \phi_j\right) = \sqrt{\frac{(2\pi)^N}{\prod_q \lambda_q}} = \sqrt{\frac{(2\pi)^N}{\det(\mathbf{A})}}\,.$$

(b) Using the result from (a), show that you can write:

$$e^{-\beta\mathcal{H}} = C\int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \, \exp\left(\frac{\beta}{2}\sum_{i,j} [-\phi_i J_{ij} \phi_j + s_i J_{ij} s_j]\right)\,,$$

where the constant $C = \sqrt{\det(\beta\mathbf{J})/(2\pi)^N}$.

<u>**Answer:**</u>

$$\begin{aligned}
e^{-\beta\mathcal{H}} &= \exp\left(\frac{\beta}{2}\sum_{i,j} s_i J_{ij} s_j\right) \\
&= \exp\left(\frac{\beta}{2}\sum_{i,j} s_i J_{ij} s_j\right)\sqrt{\frac{\det(\beta\mathbf{J})}{(2\pi)^N}}\int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \, \exp\left(-\frac{\beta}{2}\sum_{i,j} \phi_i J_{ij} \phi_j\right) \\
&= \sqrt{\frac{\det(\beta\mathbf{J})}{(2\pi)^N}}\int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \, \exp\left(\frac{\beta}{2}\sum_{i,j}[-\phi_i J_{ij} \phi_j + s_i J_{ij} s_j]\right)
\end{aligned}$$

(c) Introduce new variables $m_i = \phi_i + s_i$ for each i. (The range of integration for m_i remains the same as for ϕ_i: from $-\infty$ to ∞). Show that the integral from part (b) becomes:

$$e^{-\beta\mathcal{H}} = C\int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(\frac{\beta}{2}\sum_{i,j} [-m_i J_{ij} m_j + 2 s_i J_{ij} m_j]\right)\,.$$

<u>**Answer:**</u> For this change of variables the Jacobian has determinant 1, so $\prod_i d\phi_i = \prod_i dm_i$. The integral becomes:

$$\begin{aligned}
e^{-\beta\mathcal{H}} &= C\int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(\frac{\beta}{2}\sum_{i,j}[-(m_i - s_i) J_{ij} (m_j - s_j) + s_i J_{ij} s_j]\right) \\
&= C\int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(\frac{\beta}{2}\sum_{i,j}[-m_i J_{ij} m_j + s_i J_{ij} m_j + m_i J_{ij} s_j]\right)
\end{aligned}$$

Since $J_{ij} = J_{ji}$, we know that $\sum_{i,j} m_i J_{ij} s_j = \sum_{i,j} s_i J_{ij} m_j$. Thus we can write:

$$e^{-\beta\mathcal{H}} = C\int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(\frac{\beta}{2}\sum_{i,j}[-m_i J_{ij} m_j + 2 s_i J_{ij} m_j]\right)$$

(d) By taking the sum over all configurations $\{s_i\}$, show that you can now express the partition function Z entirely in terms of an integral over the $\{m_i\}$:

$$Z = C \int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(-\frac{\beta}{2} \sum_{i,j} m_i J_{ij} m_j + \sum_i \ln\left(2\cosh \sum_j \beta J_{ij} m_j \right) \right)$$

Hint: Use the fact that,

$$\sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \cdots \sum_{s_N=\pm 1} e^{\sum_i f(s_i)} = \prod_i \left(\sum_{s_i=\pm 1} e^{f(s_i)} \right),$$

where $f(s_i)$ is some function of s_i.

Answer: Let us denote the sum over all possible spin states as

$$\sum_{\{s_i=\pm 1\}} \equiv \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \cdots \sum_{s_N=\pm 1}$$

Then we have

$$\begin{aligned}
Z &= \sum_{\{s_i=\pm 1\}} e^{-\beta \mathcal{H}} \\
&= C \sum_{\{s_i=\pm 1\}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(\frac{\beta}{2} \sum_{i,j} [-m_i J_{ij} m_j + 2 s_i J_{ij} m_j] \right) \\
&= C \int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(-\frac{\beta}{2} \sum_{i,j} m_i J_{ij} m_j \right) \sum_{\{s_i=\pm 1\}} \exp\left(\sum_i \sum_j \beta s_i J_{ij} m_j \right) \\
&= C \int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(-\frac{\beta}{2} \sum_{i,j} m_i J_{ij} m_j \right) \prod_i \sum_{s_i=\pm 1} \exp\left(\sum_j \beta s_i J_{ij} m_j \right) \\
&= C \int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(-\frac{\beta}{2} \sum_{i,j} m_i J_{ij} m_j \right) \prod_i 2\cosh\left(\sum_j \beta J_{ij} m_j \right) \\
&= C \int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \, \exp\left(-\frac{\beta}{2} \sum_{i,j} m_i J_{ij} m_j + \sum_i \ln\left(2\cosh \sum_j \beta J_{ij} m_j \right) \right)
\end{aligned}$$

(e) Now take the continuum limit of small lattice spacing ℓ, letting $m_i = m(\mathbf{x}_i)$, where $m(\mathbf{x})$ is a continuous function over space. Show that:

$$\sum_j \beta J_{ij} m_j = 2d\beta J m(\mathbf{x}_i) + \beta J \ell^2 \sum_{\alpha=1}^{d} \partial_\alpha^2 m(\mathbf{x}_i) + \text{higher order terms}\,.$$

Here α labels the d directions in the lattice, which have associated unit vectors $\hat{\mathbf{e}}_\alpha$. The derivative along the $\hat{\mathbf{e}}_\alpha$ direction is written as $\partial_\alpha \equiv \hat{\mathbf{e}}_\alpha \cdot \nabla$. *Hint:* For each j, write $m_j = m(\mathbf{x}_j)$ as a Taylor expansion around $\mathbf{x}_i$.

Answer: For each position $\mathbf{x}_i$, there are $2d$ nearest neighbors: $\mathbf{x}_i + \ell\hat{\mathbf{e}}_\alpha$ and $\mathbf{x}_i - \ell\hat{\mathbf{e}}_\alpha$ for $\alpha = 1, \ldots, d$. Thus we can write:

$$\sum_j \beta J_{ij} m_j = \beta J \sum_{\alpha=1}^{d} m(\mathbf{x}_i + \ell\hat{\mathbf{e}}_\alpha) + \beta J \sum_{\alpha=1}^{d} m(\mathbf{x}_i - \ell\hat{\mathbf{e}}_\alpha)$$

The Taylor expansions for $m(\mathbf{x}_i + \ell\hat{\mathbf{e}}_\alpha)$ and $m(\mathbf{x}_i - \ell\hat{\mathbf{e}}_\alpha)$ are:

$$\begin{aligned} m(\mathbf{x}_i + \ell\hat{\mathbf{e}}_\alpha) &= m(\mathbf{x}_i) + \ell\partial_\alpha m(\mathbf{x}_i) + \frac{1}{2}\ell^2\partial_\alpha^2 m(\mathbf{x}_i) + \cdots \\ m(\mathbf{x}_i - \ell\hat{\mathbf{e}}_\alpha) &= m(\mathbf{x}_i) - \ell\partial_\alpha m(\mathbf{x}_i) + \frac{1}{2}\ell^2\partial_\alpha^2 m(\mathbf{x}_i) + \cdots \end{aligned}$$

Plugging these in, the first order terms cancel and we get:

$$\sum_j \beta J_{ij} m_j = 2d\beta J m(\mathbf{x}_i) + \beta J \ell^2 \sum_{\alpha=1}^{d} \partial_\alpha^2 m(\mathbf{x}_i) + \cdots$$

(f) Use the result of (e) to write the Z integral in terms of $m(\mathbf{x}_i)$ and derivatives of $m(\mathbf{x}_i)$. In the continuum limit we can make the substitutions:

$$\sum_i \ell^d F(m(\mathbf{x}_i), \nabla m(\mathbf{x}_i), \nabla^2 m(\mathbf{x}_i), \ldots) \;\rightarrow\; \int d^d\mathbf{x}\, F[m(\mathbf{x}), \nabla m(\mathbf{x}), \nabla^2 m(\mathbf{x}), \ldots]\,,$$

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} dm_i \;\rightarrow\; \int \mathcal{D}m(\mathbf{x})\,,$$

where F is some function. Finally, use the Taylor series expansion,

$$\ln\cosh x = x^2/2 - x^4/12 + \cdots\,,$$

to write the partition function in the following functional integral form:

$$Z = C \int \mathcal{D}m(\mathbf{x}) \exp\left(-\beta \int d^d\mathbf{x} \left[\frac{r}{2} m^2(\mathbf{x}) + u m^4(\mathbf{x}) + \frac{c}{2}(\nabla m(\mathbf{x}))^2 + \text{higher order}\right]\right)\,,$$

where $(\nabla m(\mathbf{x}))^2 \equiv \sum_{\alpha=1}^{d} (\partial_\alpha m(\mathbf{x}))^2$. Write the coupling constants r, u, and c in terms of β, ℓ, d, and J. *Hint:* You may need to integrate by parts at some point to get the integral into the correct form. Also, ignore any constant terms, since these will only shift the energy levels in the system, but not change the thermodynamics.

Answer: Using the result of (e) we can write the terms in the integral as:

$$\begin{aligned} -\frac{\beta}{2}\sum_{i,j} m_i J_{ij} m_j &= -\frac{1}{2}\sum_i m(\mathbf{x}_i)\left[2d\beta J m(\mathbf{x}_i) + \beta J \ell^2 \nabla^2 m(\mathbf{x}_i) + \cdots\right] \\ &= -\frac{1}{2}\sum_i \left[2d\beta J m^2(\mathbf{x}_i) + \beta J \ell^2 m(\mathbf{x}_i)\nabla^2 m(\mathbf{x}_i) + \cdots\right] \end{aligned}$$

and (using the Taylor expansion of $\ln\cosh x$):

$$\begin{aligned}\sum_i \ln\left(2\cosh\sum_j \beta J_{ij} m_j\right) &= \sum_i \ln 2 + \sum_i \left\{\frac{1}{2}\left[2d\beta J m(\mathbf{x}_i) + \beta J \ell^2 \nabla^2 m(\mathbf{x}_i) + \cdots\right]^2\right. \\ &\qquad \left. -\frac{1}{12}\left[2d\beta J m(\mathbf{x}_i) + \beta J \ell^2 \nabla^2 m(\mathbf{x}_i) + \cdots\right]^4\right\} \\ &= \sum_i \ln 2 + \sum_i \left\{2(d\beta J)^2 m^2(\mathbf{x}_i) + 2d\beta^2 J^2 \ell^2 m(\mathbf{x}_i)\nabla^2 m(\mathbf{x}_i)\right. \\ &\qquad \left. -\frac{4}{3}(d\beta J)^4 m^4(\mathbf{x}_i) + \text{higher order}\right\}\end{aligned}$$

We ignore the constant term $\sum_i \ln 2$ since it does not affect the physics. Putting everything together the partition function now looks like:

$$Z = C\int_{-\infty}^{\infty}\prod_{i=1}^{N} dm_i \,\exp\left(-\beta\sum_i \ell^d\left[\frac{r}{2}m^2(\mathbf{x}_i) + um^4(\mathbf{x}_i) - \frac{c}{2}m(\mathbf{x_i})\nabla^2 m(\mathbf{x_i}) + \cdots\right]\right)$$

where:

$$r = 2\ell^{-d}\left(dJ - 2d^2\beta J^2\right)\,, \qquad u = \frac{4}{3}\ell^{-d}d^4\beta^3 J^4\,, \qquad c = 2\ell^{2-d}\left(2d\beta J^2 - \frac{1}{2}J\right)$$

In the continuum limit Z becomes:

$$Z = C\int \mathcal{D}m(\mathbf{x})\exp\left(-\beta\int d^d\mathbf{x}\,\left[\frac{r}{2}m^2(\mathbf{x}) + um^4(\mathbf{x}) - \frac{c}{2}m(\mathbf{x})\nabla^2 m(\mathbf{x}) + \cdots\right]\right)$$

The last step is to use integration by parts to write:

$$\int d^d\mathbf{x}\, m(\mathbf{x})\nabla^2 m(\mathbf{x}) = -\int d^d\mathbf{x}\,(\nabla m(\mathbf{x}))^2$$

where we assume the surface term is zero. Thus finally we have:

$$Z = C\int \mathcal{D}m(\mathbf{x})\exp\left(-\beta\int d^d\mathbf{x}\,\left[\frac{r}{2}m^2(\mathbf{x}) + um^4(\mathbf{x}) + \frac{c}{2}(\nabla m(\mathbf{x}))^2 + \cdots\right]\right)$$

In conclusion, the field theory you have constructed has the same form as the one shown in class for an $n = 1$ component order parameter $m(\mathbf{x})$, up to the constant C in front of the path integral which does not affect the physics. In addition, for the case of the Ising model you have shown how the coupling constants in the field theory depend on parameters of the microscopic Hamiltonian like ℓ and J.