RG Methods in Statistical Field Theory: Final Exam

due: Monday, January 22, 2007

Up to now we have mainly considered phase transitions and critical phenomena driven by changes in temperature: we had a finite critical point T_c below which the system spontaneously ordered. Long-wavelength fluctuations near T_c controlled the scaling of the thermodynamic functions through the phase transition. In this problem, we will study another form of critical phenomena, associated with quantum systems at $T = 0$, where it is possible to change from one ground state to another as a function of an external parameter. Here we will need to consider both fluctuations in space and in imaginary time to fully describe the critical behavior. Our system is a dilute gas of electrons on a lattice, with a Hubbard-type interaction U. Even though the electrons are not localized, we will see that for large enough U at $T = 0$ the system develops an overall magnetization: it becomes what is known as a Stoner ferromagnet. The problem can be broken into three parts: (I) starting with the quantum Hamiltonian, we will construct a field theory in terms of the magnetization, similar in form to the Landau-Ginzburg functional; (II) we will derive the phase transition using the mean-field approximation; (III) we will get a more accurate understanding of the critical behavior by applying RG.

Part I: Construction of the field theory

We start with a Hamiltonian $\mathcal{H}_0$ for non-interacting electrons on an N-site d-dimensional lattice. In Problem Set 10 we have already derived the diagonalized form of this Hamiltonian:

$$\mathcal{H}_0 = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$$

where $\xi_{\mathbf{k}} = E_{\mathbf{k}} - \mu$, and the index $\sigma = \pm 1$ denotes up or down spin (along the z-axis). For later convenience σ has a numerical value, though we will also sometimes use the $\uparrow$, $\downarrow$ notation like in previous problems, with the understanding that $\uparrow = 1$, $\downarrow = -1$. We saw in class that for very low fillings (a small number of electrons compared to the total number of lattice sites) the energy $E_{\mathbf{k}}$ for an electron with momentum $\mathbf{k}$ can be approximately written as $E_{\mathbf{k}} \approx \mathbf{k}^2/2m$, with an effective mass $m = 1/2t$, where t is the hopping coefficient.

(a) To this $\mathcal{H}_0$ we will add an interaction term $\mathcal{H}_I$ of the Hubbard form:

$$\mathcal{H}_I = U \sum_i n_{i\uparrow} n_{i\downarrow}$$

where the sum runs over lattice sites i and the number operator $n_{i\sigma} = c^\dagger_{i\sigma} c_{i\sigma}$. Show that $\mathcal{H}_I$ can be rewritten as:

$$\mathcal{H}_I = \frac{U}{4} \sum_i n_i^2 - U \sum_i (S_i^z)^2$$

where $n_i = n_{i\uparrow} + n_{i\downarrow}$ is the particle density, and $S_i^z = \frac{1}{2}(n_{i\uparrow} - n_{i\downarrow})$ is the z-component of the spin, at site i. (Remember $\hbar = 1$.)

Thus the interaction really involves two effects: a coupling to density n_i, and a coupling to spin S_i^z. For simplicity, we assume that in a system at low filling the density coupling will have little influence on the thermodynamic properties, so we will ignore it and write $\mathcal{H}_I \approx -U \sum_i (S_i^z)^2$.

(b) The partition function Z for $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ has the form:

$$Z = \int e^S \, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi, \qquad S = \int_0^\beta \left(-\bar{\psi}(\tau) \cdot \frac{\partial}{\partial \tau} \psi(\tau) - \mathcal{H}[\bar{\psi}(\tau), \psi(\tau)] \right)$$

where $\bar{\psi}(\tau)$, $\psi(\tau)$ are Grassmann vectors. Show that the action S can be written as:

$$S = \int_0^\beta d\tau \left(\sum_{\mathbf{k},\sigma} \bar{\psi}_{\mathbf{k}\sigma}(\tau) \left[-\frac{\partial}{\partial \tau} - \xi_{\mathbf{k}} \right] \psi_{\mathbf{k}\sigma}(\tau) + \frac{U}{4} \sum_i \left[\sum_\sigma \sigma \bar{\psi}_{i\sigma}(\tau) \psi_{i\sigma}(\tau) \right]^2 \right)$$

Here $\psi_{\mathbf{k}\sigma}(\tau)$, $\bar{\psi}_{\mathbf{k}\sigma}(\tau)$ are the Grassmann vector components corresponding to $c_{\mathbf{k}\sigma}$, $c^\dagger_{\mathbf{k}\sigma}$, and $\psi_{i\sigma}(\tau)$, $\bar{\psi}_{i\sigma}(\tau)$ are the Grassmann vector components corresponding to $c_{i\sigma}$, $c^\dagger_{i\sigma}$. (For now it is simpler to write the action using both momentum-space and position-space components. Later on we will switch entirely to momentum space.)

(c) The action in part (b) is difficult to integrate directly because the U interaction term is quartic. We can simplify it by introducing a real local field $m_i(\tau)$ at each site i, and using a Hubbard-Stratonovich transformation similar to one described in Problem Set 10, part (g). The transformation is based on the following result:

$$\exp\left(\frac{1}{4a} \int d\tau \, h_i^2(\tau) \right) \propto \int \mathcal{D}m_i \, \exp\left(\int d\tau \, \left[-a m_i^2(\tau) - h_i(\tau) m_i(\tau) \right] \right)$$

where $\mathrm{Re}(a) > 0$ and $h_i(\tau)$ is an arbitrary function. Show that the partition function Z can be written as:

$$Z = \int e^S \, \mathcal{D}m \, \mathcal{D}\bar{\psi} \, \mathcal{D}\psi$$

where:

$$S = \int_0^\beta d\tau \left(-\frac{U}{4} \sum_i m_i^2(\tau) + \sum_{\mathbf{k},\sigma} \bar{\psi}_{\mathbf{k}\sigma}(\tau) \left[-\frac{\partial}{\partial \tau} - \xi_{\mathbf{k}} \right] \psi_{\mathbf{k}\sigma}(\tau) - \frac{U}{2} \sum_{i,\sigma} \sigma m_i(\tau) \bar{\psi}_{i\sigma}(\tau) \psi_{i\sigma}(\tau) \right)$$

Here $\mathcal{D}m \equiv \prod_i \mathcal{D}m_i$. Note that since $\bar{\psi}_{i\sigma}(0)\psi_{i\sigma}(0) = \bar{\psi}_{i\sigma}(\beta)\psi_{i\sigma}(\beta)$, the local fields $m_i(\tau)$ satisfy periodic boundary conditions $m_i(0) = m_i(\beta)$.

(d) Let us do two Fourier transformations in the action: convert everything that depends on site i into momentum-space form; then convert everything that depends on τ into Matsubara frequency form. The Fourier expansions are defined as follows:

$$m_i(\tau) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \sum_n e^{i\mathbf{k}\cdot\mathbf{x}_i} e^{-i\nu_n \tau} m_{\mathbf{k}n}$$

$$\psi_{i\sigma}(\tau) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \sum_{n} e^{i\mathbf{k}\cdot\mathbf{x}_i} e^{-i\omega_n \tau} \psi_{\mathbf{k}n\sigma}, \qquad \psi_{\mathbf{k}\sigma}(\tau) = \sum_{n} e^{-i\omega_n \tau} \psi_{\mathbf{k}n\sigma}$$
$$\bar{\psi}_{i\sigma}(\tau) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \sum_{n} e^{-i\mathbf{k}\cdot\mathbf{x}_i} e^{i\omega_n \tau} \bar{\psi}_{\mathbf{k}n\sigma}, \qquad \bar{\psi}_{\mathbf{k}\sigma}(\tau) = \sum_{n} e^{i\omega_n \tau} \bar{\psi}_{\mathbf{k}n\sigma}$$

Since $\psi_{i\sigma}(\tau)$ and $\bar{\psi}_{i\sigma}(\tau)$ satisfy antiperiodic boundary conditions, the ω_n are fermionic Matsubara frequencies, $\omega_n = (2n+1)\pi/\beta$. On the other hand, $m_i(\tau)$ satisfies periodic boundary conditions, so the ν_n are bosonic Matsubara frequencies, $\nu_n = 2n\pi/\beta$. Note that $m_i(\tau)$ is a real field, $m_i^*(\tau) = m_i(\tau)$, so the Fourier components are related by $m^*_{\mathbf{k}n} = m_{-\mathbf{k},-n}$. Show that the action S becomes:

$$S = -\frac{\beta U}{4} \sum_{\mathbf{k},n} m_{\mathbf{k}n} m_{-\mathbf{k},-n} + \sum_{\mathbf{k},n,\mathbf{k}',n',\sigma} \bar{\psi}_{\mathbf{k}n\sigma} \left[\beta \left(i\omega_n - \xi_{\mathbf{k}}\right) \delta_{\mathbf{k},\mathbf{k}'} \delta_{n,n'} - \frac{\beta U}{2\sqrt{N}} \sigma m_{\mathbf{k}-\mathbf{k}',n-n'} \right] \psi_{\mathbf{k}'n'\sigma}$$

(e) The second sum in the expression for S above has the form of a matrix sandwiched between two vectors. Let $\bar{\psi}_\sigma$ and ψ_σ denote Grassmann vectors with components labeled by $(\mathbf{k}, n)$. For example, the $(\mathbf{k}, n)$th component of ψ_σ is $\psi_{\mathbf{k}n\sigma}$. Let M^σ denote a matrix with components $M^\sigma_{\mathbf{k}n,\mathbf{k}'n'}$ given by:

$$M^\sigma_{\mathbf{k}n,\mathbf{k}'n'} = -\beta \left(i\omega_n - \xi_{\mathbf{k}}\right) \delta_{\mathbf{k},\mathbf{k}'} \delta_{n,n'} + \frac{\beta U}{2\sqrt{N}} \sigma m_{\mathbf{k}-\mathbf{k}',n-n'}$$

The index σ means that we have a different matrix, and a different set of Grassmann vectors, for each σ. M^σ can be decomposed into two parts, $M^\sigma = M_0 + M^\sigma_U$, where M_0 is a diagonal matrix, and M^σ_U is an off-diagonal part that depends on U:

$$(M_0)_{\mathbf{k}n,\mathbf{k}'n'} = -\beta \left(i\omega_n - \xi_{\mathbf{k}}\right) \delta_{\mathbf{k},\mathbf{k}'} \delta_{n,n'}, \qquad (M^\sigma_U)_{\mathbf{k}n,\mathbf{k}'n'} = \frac{\beta U}{2\sqrt{N}} \sigma m_{\mathbf{k}-\mathbf{k}',n-n'}$$

Our action then takes the form:

$$S = -\frac{\beta U}{4} \sum_{\mathbf{k},n} m_{\mathbf{k}n} m_{-\mathbf{k},-n} - \sum_{\sigma} \bar{\psi}^T_\sigma M^\sigma \psi_\sigma$$

By integrating over the Grassmann vector components, show that the partition function Z can be written:

$$Z = \int e^S \prod_{\mathbf{k},n} dm_{\mathbf{k}n}, \qquad S = -\frac{\beta U}{4} \sum_{\mathbf{k},n} m_{\mathbf{k}n} m_{-\mathbf{k},-n} + \sum_{\sigma} \ln \det M^\sigma$$

(f) We now have the partition function written in an almost classical form: an integral over the Fourier components $m_{\mathbf{k},n}$. As we will see later, we can interpret $m_{\mathbf{k},n}$ as describing a fluctuation in the magnetization along the z-axis with spatial wavevector $\mathbf{k}$ and imaginary time frequency ν_n. However, to understand the physical content of the action S we need to find a way to simplify the $\sum_\sigma \ln\det M^\sigma$ term. The first step is to prove the following identity:

$$\ln\det A = \mathrm{Tr}\,\ln A$$

for any diagonalizable matrix A. The matrix logarithm is defined through $\ln A = \ln(I + (A-I)) = (A-I) - \frac{1}{2}(A-I)^2 + \frac{1}{3}(A-I)^3 + \cdots$, where I is the identity matrix, and it satisfies $e^{\ln A} = A$. *Hint:* Since A is diagonalizable, there is some similarity transformation $SAS^{-1} = D$, where D is a diagonal matrix. Start by showing that $\ln\det D = \mathrm{Tr}\,\ln D$. Then replace D by SAS^{-1} and manipulate the expressions until you find $\ln\det A = \mathrm{Tr}\,\ln A$. Remember that $\det(XY) = \det(YX)$, $\mathrm{Tr}\,(XY) = \mathrm{Tr}\,(YX)$.

(g) Using the identity in part (f), and the fact that $\det(XY) = \det(X)\det(Y)$, prove the additional identity:

$$\mathrm{Tr}\,\ln(AB) = \mathrm{Tr}\,\ln A + \mathrm{Tr}\,\ln B$$

for any two diagonalizable matrices A, B. (A and B do not have to commute.)

(h) Let us write the matrix M^σ as $M^\sigma = M_0 + M_U^\sigma = M_0(I + M_0^{-1}M_U^\sigma) = M_0(I + K^\sigma)$, where $K^\sigma \equiv M_0^{-1}M_U^\sigma$. Using the identities in parts (f) and (g) show that Z can be written as:

$$Z = C_0 \int e^S \prod_{\mathbf{k}n} dm_{\mathbf{k},n}, \qquad S = -\frac{\beta U}{4}\sum_{\mathbf{k},n} m_{\mathbf{k}n}m_{-\mathbf{k},-n} + \sum_\sigma \mathrm{Tr}\,\ln(I + K^\sigma)$$

where C_0 is a constant independent of all $m_{\mathbf{k},n}$.

(i) Show that the components of the matrix K^σ have the form:

$$K^\sigma_{\mathbf{k}n,\mathbf{k}'n'} = -\frac{U\sigma}{2\sqrt{N}}\frac{m_{\mathbf{k}-\mathbf{k}',n-n'}}{i\omega_n - \xi_\mathbf{k}} = -\frac{U\sigma}{2\sqrt{N}} m_{\mathbf{k}-\mathbf{k}',n-n'} G(\mathbf{k},n)$$

where $G(\mathbf{k},n) \equiv (i\omega_n - \xi_\mathbf{k})^{-1}$.

We want to construct a field theory near the phase transition, where the magnetization components $m_{\mathbf{k}n}$ are small, and since the components of the K^σ matrix are proportional to the magnetization, we will expand $\mathrm{Tr}\,\ln(I + K^\sigma)$ in a Taylor series, keeping only terms up to fourth order:

$$\mathrm{Tr}\,\ln(I + K^\sigma) = \mathrm{Tr}\,K^\sigma - \frac{1}{2}\mathrm{Tr}\,[(K^\sigma)^2] + \frac{1}{3}\mathrm{Tr}\,[(K^\sigma)^3] - \frac{1}{4}\mathrm{Tr}\,[(K^\sigma)^4] + \cdots$$

(j) Show that for odd powers m,

$$\sum_\sigma \mathrm{Tr}\,[(K^\sigma)^m] = 0$$

so that only the even powers in the expansion contribute to S:

$$S = -\frac{\beta U}{4}\sum_{\mathbf{k},n} m_{\mathbf{k}n}m_{-\mathbf{k},-n} - \frac{1}{2}\sum_\sigma \mathrm{Tr}\,[(K^\sigma)^2] - \frac{1}{4}\sum_\sigma \mathrm{Tr}\,[(K^\sigma)^4] + \cdots$$

(k) By writing out the matrix multiplication and the trace explicitly, and through careful relabeling of the indices, show that the two K^σ terms in the action can be written as:

$$\frac{1}{2}\sum_\sigma \mathrm{Tr}\,[(K^\sigma)^2] = \sum_{\mathbf{k},n} v_2(\mathbf{k},n) m_{\mathbf{k}n} m_{-\mathbf{k},-n}$$

$$\frac{1}{4}\sum_\sigma \mathrm{Tr}\,[(K^\sigma)^4] =$$

$$\sum_{\mathbf{k}_1,n_1,\mathbf{k}_2,n_2,\mathbf{k}_3,n_3} v_4(\mathbf{k}_1,n_1;\mathbf{k}_2,n_2;\mathbf{k}_3,n_3) m_{\mathbf{k}_1 n_1} m_{\mathbf{k}_2 n_2} m_{\mathbf{k}_3 n_3} m_{-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3,-n_1-n_2-n_3}$$

where:

$$v_2(\mathbf{k},n) = \frac{U^2}{4N}\sum_{\mathbf{q},m} G(\mathbf{q},m)G(\mathbf{q}+\mathbf{k},m+n)$$

$$v_4(\mathbf{k}_1,n_1;\mathbf{k}_2,n_2;\mathbf{k}_3,n_3) = \frac{U^4}{32N^2}\sum_{\mathbf{q},m} G(\mathbf{q},m)G(\mathbf{q}+\mathbf{k}_3,m+n_3)G(\mathbf{q}+\mathbf{k}_2+\mathbf{k}_3,m+n_2+n_3)$$
$$\cdot\, G(\mathbf{q}+\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3,m+n_1+n_2+n_3)$$

In principle it is possible to do the Matsubara frequency sum and evaluate explicitly the function $v_2(\mathbf{k},n)$. However this involves a tedious calculation and we are only interested in the long wavelength, small frequency behavior near $T=0$, since this will determine the phase transition properties. The end result (which you do not have to derive) looks like:

$$v_2(\mathbf{k},n) \approx -\frac{\beta U^2\rho(E_F)}{4}\left(1 - \frac{Ak^2}{k_F^2} - \frac{B|\nu_n|}{v_F k} + \cdots\right)$$

where $k = |\mathbf{k}|$, and A, B are numerical constants depending on dimension d. In $d=3$, for example, $A = 1/12$ and $B = \pi/2$. Here $\rho(E_F)$ is the density of states at the Fermi surface (see Problem Set 11), k_F is the Fermi momentum defined through $k_F^2/2m = E_F$, and $v_F = k_F/m$ is the Fermi velocity. The presence of quantities like $\rho(E_F)$, k_F, and v_F makes sense: recall that the thermodynamic properties of an electron gas near $T=0$ are controlled by modes around the Fermi surface. In fact we saw in class that $E_\mathbf{k} - E_F \approx v_F k$ for $k \approx k_F$, so $v_F k$ is just the excitation energy of an electron near the Fermi surface.

The function v_4 is even more complicated, but its nontrivial momentum and frequency dependence does not affect the critical behavior. We make the simplest approximation and just replace it with a constant, writing:

$$\frac{1}{4}\sum_\sigma \mathrm{Tr}\,[(K^\sigma)^4] \approx \frac{\beta u}{N}\sum_{\mathbf{k}_1,n_1,\mathbf{k}_2,n_2,\mathbf{k}_3,n_3} m_{\mathbf{k}_1 n_1} m_{\mathbf{k}_2 n_2} m_{\mathbf{k}_3 n_3} m_{-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3,-n_1-n_2-n_3}$$

Here $u > 0$ is a constant with units of energy.

(l) Putting everything together, show that you can write the action as:

$$S = -\beta \left[\frac{1}{2} \sum_{\mathbf{k},n} w(\mathbf{k}, n) m_{\mathbf{k}n} m_{-\mathbf{k},-n} + \frac{u}{N} \sum_{\substack{\mathbf{k}_1,n_1,\mathbf{k}_2,n_2,\\ \mathbf{k}_3,n_3}} m_{\mathbf{k}_1 n_1} m_{\mathbf{k}_2 n_2} m_{\mathbf{k}_3 n_3} m_{-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3,-n_1-n_2-n_3} \right]$$

where:

$$w(\mathbf{k}, n) \equiv r + ck^2 + \frac{|\nu_n|}{vk}$$

and the constants r, c, and v are given by:

$$r = \frac{U}{2}(1 - U\rho(E_F)), \qquad c = \frac{U^2 \rho(E_F) A}{2k_F^2}, \qquad v = \frac{2v_F}{U^2 \rho(E_F) B}$$

We now have an action that looks similar to a Landau-Ginzburg functional in momentum space. We have a parameter r multiplying an m^2-type term, the energy cost of spatial fluctuations is controlled by a parameter c, and we have an m^4-type interaction with coefficient u. However, there are two important differences from the classical Landau-Ginzburg functional: (1) In the classical system r was a "thermal" variable, $r \propto T - T_c$. In the quantum system, r varies with the interaction U. As we will see below, the phase transition occurs at $r = 0$, corresponding to $U = U_c = 1/\rho(E_F)$, so we can write $r \propto U_c - U$. (2) There is an additional energy cost associated with fluctuations in imaginary time, described by the parameter v, which was not present in the classical case.

Part II: Mean-field theory

(m) Since fluctuations in space and imaginary time cost energy, let us at first ignore them and derive a mean-field description of the phase transition. We assume the magnetization is a uniform function $m_i(\tau) = \bar{m}$ at every site i and time τ, where $\bar{m}$ is some constant. Show that the corresponding Fourier-transformed magnetization is:

$$m_{\mathbf{k}n} = \sqrt{N} \bar{m}\, \delta_{\mathbf{k},0} \delta_{n,0}$$

Plug this into the action from part (l), and then maximize the action with respect to $\bar{m}$. Show that S is maximum at $\bar{m} = 0$ for $r > 0$, and at $\bar{m} = \sqrt{-r/4u}$ when $r < 0$. Thus there is a transition into a phase with nonzero magnetization when we increase the interaction strength U above a critical value $U_c = 1/\rho(E_F)$.

Part III: Renormalization-group analysis

Before deriving the renormalization-group flows for this system, let us rewrite the action to make it more suitable for RG analysis. First we transform the sums over momenta into integrals:

$$\sum_{\mathbf{k}} \to V \int_0^{\Lambda} \frac{d^d \mathbf{k}}{(2\pi)^d}$$

where V is the volume of the system and we have introduced a momentum-space cutoff Λ. The sums over Matsubara frequencies we will keep as sums, but we will also introduce a frequency cutoff Γ. Since $\nu_n = 2n\pi/\beta = 2n\pi T$, this means that the sums will run over all n satisfying $|n| < n_{\max} \equiv \Gamma/2\pi T$. (We are using units where $k_B = 1$.) We do not lose any interesting physics by introducing these cutoffs, since it is the long wavelength, low frequency modes that we expect to be dominant at the phase transition. Our action now has the form $S = S_0 + S_I$, where:

$$S_0 = -\frac{V}{2T}\int_0^\Lambda \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_{|n|<\Gamma/2\pi T} \left(r + ck^2 + \frac{|\nu_n|}{vk}\right) m_{\mathbf{k}n} m_{-\mathbf{k},-n}$$

$$S_I = -\frac{uV^3}{NT}\int_0^\Lambda \frac{d^d\mathbf{k}_1 \cdots d^d\mathbf{k}_4}{(2\pi)^{3d}} \sum_{|n_1|,\dots,|n_4|<\Gamma/2\pi T} m_{\mathbf{k}_1 n_1} \cdots m_{\mathbf{k}_4 n_4}\, \delta^{(d)}(\mathbf{k}_1 + \cdots + \mathbf{k}_4)\delta_{n_1+\cdots+n_4,0}$$

We can further simplify the action by choosing units such that $V/N = \ell^d = 1$, and by absorbing a factor of $\sqrt{V}$ into each $m_{\mathbf{k}n}$. Then we have:

$$S_0 = -\frac{1}{2T}\int_0^\Lambda \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_{|n|<\Gamma/2\pi T} \left(r + ck^2 + \frac{|\nu_n|}{vk}\right) m_{\mathbf{k}n} m_{-\mathbf{k},-n}$$

$$S_I = -\frac{u}{T}\int_0^\Lambda \frac{d^d\mathbf{k}_1 \cdots d^d\mathbf{k}_4}{(2\pi)^{3d}} \sum_{|n_1|,\dots,|n_4|<\Gamma/2\pi T} m_{\mathbf{k}_1 n_1} \cdots m_{\mathbf{k}_4 n_4}\, \delta^{(d)}(\mathbf{k}_1 + \cdots + \mathbf{k}_4)\delta_{n_1+\cdots+n_4,0}$$

A contraction with respect to S_0 has the form:

$$\langle m_{\mathbf{k}n} m_{\mathbf{k}'n'} \rangle_0 = \frac{(2\pi)^d \delta^{(d)}(\mathbf{k}+\mathbf{k}')\delta_{n+n',0}}{\beta\left(r + ck^2 + \frac{|\nu_n|}{vk}\right)}$$

(n) Let us start with the "Gaussian" model for this system: set $u = 0$, so that $S = S_0$. Like the classical Gaussian model, our results will be strictly valid only for $r > 0$, the disordered phase. Renormalization proceeds in the standard manner: we integrate out the fast modes, and find an effective Hamiltonian $S_<$ in terms of the slow modes. The fast modes in this case will be all $m_{\mathbf{k}n}$ with $\mathbf{k}$ inside the shell $\Lambda/b < k < \Lambda$ or n inside the shell $\Gamma/b^z < |\nu_n| < \Gamma$. Note that we rescale the frequency cutoff Γ by a factor b^z, which can in general be different than the rescaling factor b of the momentum cutoff (if $z \neq 1$). The usefulness of different rescaling factors will become apparent below. z is known as the *dynamic exponent*, and we will choose it later. Thus we define:

$$m_{\mathbf{k}n} = \begin{cases} m_{\mathbf{k}n<} & \text{if } k < \Lambda/b \text{ and } |\nu_n| < \Gamma/b^z \\ m_{\mathbf{k}n>} & \text{if } \Lambda/b < k < \Lambda \text{ or } \Gamma/b^z < |\nu_n| < \Gamma \end{cases}$$

Integrate out the fast modes, and show that the effective slow-mode Hamiltonian $S_<$ has the form:

$$S_< = -\frac{1}{2T}\int_0^{\Lambda/b} \frac{d^d\mathbf{k}}{(2\pi)^d} \sum_{|n|<b^{-z}\Gamma/2\pi T} \left(r + ck^2 + \frac{|\nu_n|}{vk}\right) m_{\mathbf{k}n<} m_{-\mathbf{k},-n<}$$

To get $S_<$ back in the same form as S, we introduce renormalized variables $\mathbf{k}' = b\mathbf{k}$, $\nu_n' = b^z \nu_n$, and $m_{\mathbf{k}'n'} = \zeta^{-1} m_{\mathbf{k}n<}$. Since $\nu_n \propto T$, renormalized frequencies imply a renormalized temperature $T' = b^z T$. Show that $S_<$ can be written as:

$$S' = -\frac{1}{2T'} \int_0^\Lambda \frac{d^d\mathbf{k}'}{(2\pi)^d} \sum_{|n|<\Gamma/2\pi T'} \left(r' + c'k'^2 + \frac{|\nu_n'|}{v'k'} \right) m'_{\mathbf{k}'n'} m'_{-\mathbf{k}',-n'}$$

Find expressions for r', c', and v'. Just like in the Landau-Ginzburg case, we will choose our RG transformation to fix the coefficients of certain terms: in this case, choose ζ and z so that $c' = c$, $v' = v$, since we know that c and v are both positive constants in the original system, setting the scales for spatial and imaginary time fluctuations. You should find that $z = 3$.

(o) The RG equations for T and r from part (n) look like:

$$T' = b^z T, \qquad r' = b^2 r$$

with $z = 3$. There exists a quantum Gaussian fixed point at $T^* = r^* = 0$, controlling the critical behavior as you approach the transition point at $T = 0$ from the disordered side, $r \to 0^+$. However, for any finite temperature $T > 0$, we flow away from the quantum Gaussian fixed point toward higher temperatures, where the assumptions underlying our derivation break down, and we will eventually recover classical critical behavior. However, if the flow away from $T = 0$ is sufficiently slow then the critical behavior still looks "quantum". To determine the condition for this, let us start near the phase transition at a small value of $r > 0$ and a finite T. What is the magnitude of b required to move far away from the fixed point, for example $r' \approx 1$? At this value of b, what is the renormalized temperature $T'(b)$? If $T'(b)$ is small, $T'(b) \ll 1$, then we can say the critical scaling of the system will be controlled by the quantum fixed point. Show that this is true for $T \ll r^{z/2}$. Thus as r gets smaller, we need to go to lower temperatures to see quantum critical behavior.

(p) Finally, let us include the interaction part S_I in the action: $S = S_0 + S_I$. Work out the RG equations to first order in the cumulant expansion for an infinitesimal rescaling factor $b = e^{\delta l} \approx 1 + \delta l$. Show that the flow equations take the form:

$$T' = T + \frac{dT}{dl}\delta l, \qquad r' = r + \frac{dr}{dl}\delta l, \qquad u' = u + \frac{du}{dl}\delta l$$

where:

$$\begin{aligned} \frac{dT}{dl} &= zT \\ \frac{dr}{dl} &= 2r + 6uf(T) \\ \frac{du}{dl} &= (4 - d - z)u \end{aligned}$$

Here $f(T)$ is a complicated function of temperature for which you should write down an expression. You do not have to evaluate the integrals or sums (unless the integral is over an infinitesimal shell, in which case it can be approximated by the volume of the shell times the value at the shell edge). *Hint:* The first order in the cumulant expansion, $\langle S_I \rangle_{0>}$, consists of two diagrams: a four-vertex with two legs contracted together (contributing to r'), and a four-vertex with no contracted legs (contributing to u'). Since T is small, you may find it convenient to express sums over Matsubara frequencies ν_n as integrals:

$$\sum_n \to \frac{\beta}{2\pi} \int d\nu$$

Also remember that the fast mode region really consists of two pieces: (1) the modes where $\Lambda/b < k < \Lambda$, $0 < |\nu_n| < \Gamma$; (2) the modes where $0 < k < \Lambda$, $\Gamma/b^z < |\nu_n| < \Gamma$. There is a small overlap between these two regions, but you can ignore it since b is close to 1. If you have a function $F(\mathbf{k}, |\nu|)$ which you need to integrate over the fast modes, you can thus write result as follows:

$$\int_> F(\mathbf{k}, |\nu|) = \int_{\Lambda/b}^{\Lambda} d^d\mathbf{k}\, 2 \int_0^{\Gamma} d\nu\, F(\mathbf{k}, |\nu|) + \int_0^{\Lambda} d^d\mathbf{k}\, 2 \int_{\Gamma/b^z}^{\Gamma} d\nu\, F(\mathbf{k}, |\nu|)$$

The factors of 2 come from summing over positive and negative ν. Other factors like $(2\pi)^d$ are not shown here for simplicity.

(q) Check that the Gaussian solution $T^* = r^* = u^* = 0$ is still a fixed point of the RG equations derived in part (p). Find the dimension d_c above which the u direction is irrelevant. In other words, above this upper critical dimension d_c the phase transition behavior is controlled by the quantum Gaussian fixed point even for small $u \neq 0$. For $d < d_c$ adding the interaction term to the system will take you to another fixed point (not seen in these equations, but apparent at higher orders in the cumulant expansion), with non-Gaussian critical behavior. While $d_c = 4$ for the classical Landau-Ginzburg model, you should find in the quantum system that d_c depends on the dynamic exponent z: this shows directly the influence which imaginary time fluctuations have on quantum critical behavior.