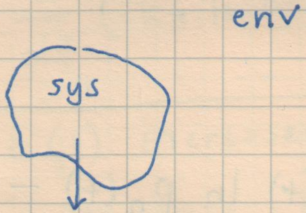
Extensivity of entropy

Look closer at $S(t) \to S_{max} = k_B \ln N$ for a system with all $E_n = E$.



state $n = 1, \ldots, N$

energy $E_1 = E_2 = \cdots = E_N$

transitions from $n \to m$ involve no net exchange of energy w/ environment, (or particle number, etc.)

[does not rule out interactions of some kind]

No net exchange can sometimes be achieved by assuming large, homogeneous system, i.e.

box of gas with many particles

imaginary volume boundary

approximate constants:

$\overset{E_A}{\overset{\shortparallel}{\text{energy of A}}}$, $\overset{E_B}{\overset{\shortparallel}{\text{energy of B}}}$, (up to negligible fluctuations if # part. is large)

particles A, # particles B,

etc.

A can be in states $n = 1, \ldots, N$ (all same energy)

B can be in states $q = 1, \ldots, M$ (all same energy)

Since boundary interactions few compared to bulk, can assume state of A indep. of state of B

joint probability $p_{n,q}(t) = p_n(t)\, p_q(t)$ if indep.

joint entropy $S(t) = -k_B \sum_{n,q} p_n p_q \ln p_n p_q$

$$= -k_B \sum_{n,q} p_n p_q \ln p_n$$

$$- k_B \sum_{n,q} p_n p_q \ln p_q$$

$$= -k_B \sum_n p_n \ln p_n$$

$$- k_B \sum_q p_q \ln p_q$$

$$= S_A(t) + S_B(t)$$

entropy is additive (extensive) for independent systems

In this case both inc. independently

$$S_A(t) \to k_B \ln N \quad t \to \infty$$
$$S_B(t) \to k_B \ln M \quad t \to \infty$$

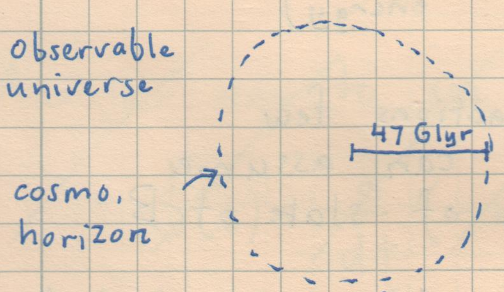
(of course additivity is an approximation, dependent on independence)

joint energy is also additive:

$$E(t) = \sum_{n,q} p_{n,q}(t)\,(E_A + E_B)$$
$$= E_A + E_B$$

So entropy of a composite object (i.e. universe) can be approx. broken down into additive components.

Entropy of universe : arXiv: 0909.3983
0801.1847



use comoving coordinates (factoring out expansion)
= current proper distance
= 47 billion light years

large scale homogeneity: no net energy, particle, etc. flows across boundary

$\Rightarrow$ $S(t)$ of ~~whole a~~ observable universe increasing toward S_{max}

What contributes to $S(t)$? (assuming additivity)

budget:
- stars (and us): $10^{81}\,k_B$
- photons : $10^{89}\,k_B$
- dark matter, gravitons, neutrinos : $\sim 10^{87} - 10^{89}\,k_B$
- all stellar black holes: $10^{97}\,k_B$ $(2.5 - 15\,M_\odot)$

(single sm BH: $\sim 10^{91}\, k_B$)

all supermassive black holes: $10^{104}\, k_B$

total: $\approx 10^{104}\, k_B$

To good approximation, virtually all entropy in obs. universe is in supermassive black holes.

In fact, a black hole is the most entropy dense volume possible in physics (most microstates per unit volume).

Bekenstein-Hawking entropy of black hole

$$S_{BH} = \frac{k_B A_{BH}}{4 \ell_p^2} \propto M_{BH}^2$$

A_{BH}: area of event horizon

M_{BH}: BH mass

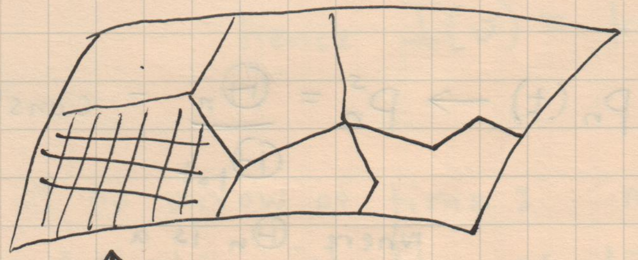
$\ell_p = 1.6 \times 10^{-35}$ m $= \sqrt{\frac{\hbar G}{c^3}}$

We can then say $S_{max} < \dfrac{k_B A_{obs.\ univ.}}{4 \ell_p^2} \approx 10^{123}\, k_B$

	early universe (after inflation)	now	heat death
S:	$\approx 10^{88}\, k_B$ (radiation dominated)	$\approx 10^{104}\, k_B$	$< 10^{123}\, k_B$

Heat death:
- if proton decay occurs, all baryonic matter $\Rightarrow$ photons + leptons
- by 10^{40} yrs, only s.m. BH left to dominate
- largest SMBH evaporates by Hawking radiation in $\sim 10^{100}$ years

Concept of temperature



se II: E_n are different

Consider case where sys exchanges energy w/ environment, so E_{tot} is conserved.

$$\frac{W_{nm}}{W_{mn}} = \frac{\Theta(E_{tot}-E_n)}{\Theta(E_{tot}-E_m)} \quad \text{detailed balance}$$

previously: all E_n are the same $\Rightarrow$ W symmetric

now: E_n are different $\Rightarrow$ W asymmetric

Additional assumptions: • system energies $E_n \ll E_{tot}$ for all n (environment is big)

+ Θ is a continuous function of energy

$$\cancel{\Rightarrow \Theta(E_{tot}-E_n) \approx \Theta(E_{tot}) - \frac{\partial \Theta}{\partial E}(E_{tot})\, E_n + \cdots}$$

Rewrite det. balance as:

$$\frac{W_{nm}}{W_{mn}} = \exp\left[\ln \Theta(E_{tot}-E_n) - \ln \Theta(E_{tot}-E_m)\right]$$

Taylor series:

$$\ln \Theta(E_{tot}-E_n) \approx \ln \Theta(E_{tot}) - \frac{\partial \ln \Theta}{\partial E}(E_{tot})\, E_n + \cdots$$

$$\Rightarrow \quad \frac{W_{nm}}{W_{mn}} = \exp\left[- \underbrace{\frac{\partial \ln \Theta}{\partial E}(E_{tot})}_{= \text{ constant}} (E_n - E_m) \right]$$

$$\equiv \beta \equiv \frac{1}{k_B T}$$

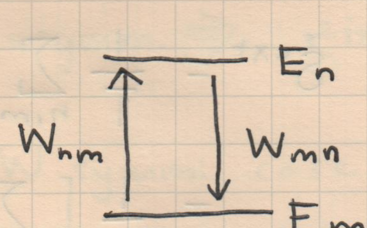
where T is the "temperature" of the environment around system (also known as a <u>thermal heat bath</u>)

$$\Rightarrow T = \frac{\partial}{\partial E} \Big[\underbrace{k_B \ln \Theta(E_{tot})}_{\substack{\text{equilibrium} \\ \text{entropy of the} \\ \text{environment w/o} \\ \text{system}}} \Big]$$

$\doteq$ positive or negative (most common)

We can interpret T as a measure of how willing the environment is to lend system energy:

Imagine $E_n > E_m$:
$T > 0$



$$\frac{W_{nm}}{W_{mn}} = \frac{\text{uphill transition}}{\text{downhill transition}} = e^{-\beta(E_n - E_m)}$$

the bigger T is, the smaller $\beta \Rightarrow$ uphill becomes relatively more likely

$$\frac{W_{nm}}{W_{mn}} = e^{-\beta(E_n - E_m)}$$

$$0 = \frac{dp_n^s}{dt} = \sum_m{}' \left(W_{nm} p_m^s - W_{mn} p_n^s \right)$$

$$= \sum_m W_{mn} \left(e^{-\beta(E_n - E_m)} p_m^s - p_n^s \right)$$

solution:

$$\boxed{p_m^s = \frac{e^{-\beta E_m}}{Z}}$$

$$Z = \sum_n{}' \cancel{p_n^s} \, e^{-\beta E_n}$$

Boltzmann equilibrium distribution (canonical ensemble)

for normalization

$$D_{KL}(\vec{p}(t) \,||\, \vec{p}^s) = \sum_n{}' p_n(t) \ln p_n(t) - \sum_n p_n \ln \left[Z^{-1} e^{-\beta E_n} \right]$$

$$= -\frac{S(t)}{k_B} + \ln Z + \beta \underbrace{\sum_n p_n(t) E_n}_{\equiv \bar{E}(t)}$$

avg. energy

$$\equiv \frac{1}{k_B T}\left[F(t) - F_{min} \right] \geq 0$$

where $F(t) = \bar{E}(t) - T S(t) =$ Helmholtz free energy

$F_{min} = -k_B T \ln Z \rightarrow$ minimum (equilibrium) Helmh. free energy

Since $\frac{d}{dt} D_{KL}(\vec{p}(t) \,||\, \vec{p}^{\,s}) < 0$ for $\vec{p}(t) \neq \vec{p}^{\,s}$

$\Rightarrow$ $\boxed{\text{F(t) decreases monotonically until reaches } F_{min}}$

Helmholtz free energy minimization!

(another version of 2nd law!)

(here $S(t)$ does not necessarily go to S_{max})