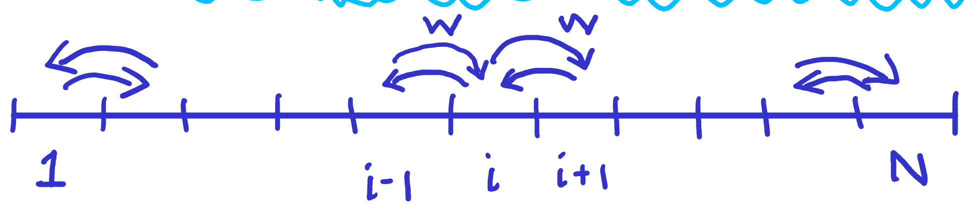
PHYS 320/420: Lecture 4



Last time: equ. for $p_i(t)$ = prob. to be at i at time t

$$1 < i < N: \quad \frac{dp_i}{dt} = -2w\, p_i(t) + w\left(p_{i+1}(t) + p_{i-1}(t)\right)$$

$$\frac{dp_1}{dt} = -w\, p_1(t) + w\, p_2(t)$$

$$\frac{dp_N}{dt} = -w\, p_N(t) + w\, p_{N-1}(t)$$

$$\vec{p}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_N(t) \end{pmatrix}$$

ith comp. of vector $\vec{p}(t) = p_i(t)$

equ. (*):

$$\frac{d\vec{p}(t)}{dt} = \underset{\text{matrix}}{\Omega}\, \vec{p}(t) \underset{\substack{i\text{th} \\ \text{comp.}}}{\Rightarrow} \frac{dp_i}{dt} = \sum_{j=1}^{N} \Omega_{ij}\, p_j(t)$$

N equations for $i = 1, \dots, N$

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_N \end{pmatrix} = \begin{pmatrix} -w & w & 0 & 0 & 0 & 0 \\ w & -2w & w & 0 & 0 & 0 \\ 0 & w & -2w & w & 0 & 0 \\ 0 & 0 & w & -2w & w & 0 \\ 0 & 0 & 0 & & \ddots & w \\ 0 & 0 & 0 & 0 & w & -w \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_N \end{pmatrix}$$

in more general terms: a set of states denoted by $i = 1, 2, \ldots, N$, and a set of transition rates

$\underline{\underline{i \neq j}}$: Ω_{ij} = prob. rate for transition from $j \to i$

$\Omega_{\underset{\text{row}}{i}\underset{\text{column}}{j}}$

universal property of Ω matrix: every column must sum to zero

Proof: $$\sum_{i=1}^{N} p_i(t) = 1 \quad \text{at every } t$$

$$\frac{d}{dt}\sum_i p_i(t) = 0$$

$$\sum_i \frac{d}{dt} p_i(t) = 0$$

note we plug in eq. (*)

$$\sum_i \sum_j \Omega_{ij}\, p_j(t) = 0$$

$$\sum_j \left[\sum_i \Omega_{ij} \right] p_j(t) = 0$$

must be true always

$1 \geq P_j(t) \geq 0$ could be anything

$\Rightarrow$ only way this could work is if $\boxed{\sum_i \Omega_{ij} = 0}$ for any column j

master equ: $\frac{dp_i}{dt} = \sum_j \Omega_{ij} p_j(t)$

$\downarrow$

can lead to useful addit. equations even if you can't directly solve it

average of $i^n \Rightarrow$ "ith moment"

$$\langle i^n \rangle_t = \sum_{i=1}^{N} i^n p_i(t)$$

$$\frac{d\langle i^n \rangle_t}{dt} = \sum_i i^n \frac{dp_i}{dt}$$ plug in eq.(*)

$$= \sum_{i,j} i^n \Omega_{ij} p_j$$

$$\sum_{i,j} = \sum_{i=1}^{N} \sum_{j=1}^{N} = \underbrace{\sum_{j=1}^{N} \sum_{i \neq j}}_{\text{off-diag elements}} + \underbrace{\sum_{j}}_{\text{diag. elements}}$$

$$\frac{d\langle i^n \rangle_t}{dt} = \sum_j \sum_{i \neq j} i^n \Omega_{ij} P_j + \sum_j j^n \Omega_{jj} P_j$$

use the property that sum of each column is zero:

$$\Omega_{jj} = -\sum_{i \neq j} \Omega_{ij}$$

$$\frac{d\langle i^n \rangle_t}{dt} = \sum_j \sum_{i \neq j} \left[i^n \Omega_{ij} P_j - j^n \Omega_{ij} P_j \right]$$

$$= \sum_j \sum_{i \neq j} (i^n - j^n) \Omega_{ij} P_j$$

eqn. just involving off-diag. elements of Ω

Digression: illustrate these ideas for simplest possible "biological" model

$N=2$

$$\underset{\text{life}}{①} \xrightarrow{w} \underset{\text{death}}{②}$$

in every time step δt you have $w\,\delta t$ prob. of dying

const. prob. of transition $\Rightarrow$ Poisson process

$$\Omega = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & -w & 0 \\ 2 & w & 0 \end{array}$$

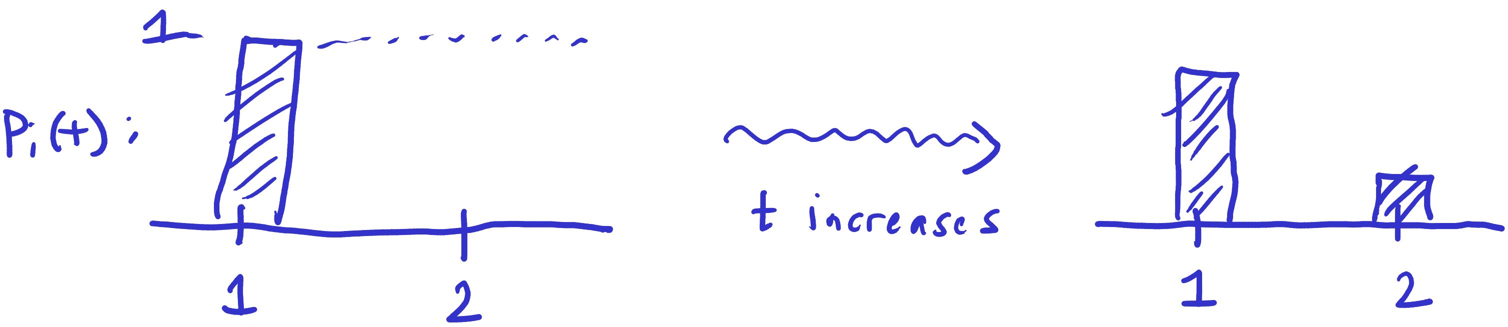
$$\frac{d}{dt}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} -w & 0 \\ w & 0 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$\frac{d}{dt} P_1 = -w P_1 \quad \Rightarrow \quad P_1(t) = P_1(0)e^{-wt}$$

$$\frac{d}{dt} P_2 = +w P_1 \qquad P_2(t) = 1 - P_1(t)$$

$$= 1 - P_1(0)\, e^{-wt}$$

$t=0$: $P_1(0)=1$



equ. for $n=1$ moment: $\langle i \rangle = \sum_{i=1}^{2} i \, P_i(t)$

$$\frac{d\langle i \rangle}{dt} = (2-1)\,\underbrace{\Omega_{21}}_{W}\, P_1(t)$$

only
$i=2$ $j=1$
element
survives

plug in $P_1(t)$ expression $\Rightarrow$ solve for $\langle i \rangle_t$

$$\langle i \rangle_t = 2 - e^{-wt} P_1(0)$$